Question

If $z = x + iy$, $w = u + iv$, and $z = w^3$ calculate x and y as functions of u and v. Substitute these expressions into the two functions of question 1. Hence confirm that those functions are harmonic in the w-plane, under the transformation $z = w^3$.

Answer

If $x = w^3$ then

$x + iy = (u + iv)^3 = (u^3 - 3uv^2) + i(3u^2v - v^3)$

Therefore $\begin{cases} x &= u^3 - 3uv^2 \\ y &= 3u^2v - v^3 \end{cases}$

Hence

(a)

$$\begin{aligned} \phi &= (u^3 - 3uv^2)^2 - (3u^2v - v^3)^2 + 2(3u^2v - v^3) \\ &= u^6 - 15u^4v^2 + 15u^2v^4 - v^6 + 6u^2v - 2v^3 \end{aligned}$$

Therefore

$\phi_{uu} = 30u^4 - 180u^2v^2 + 30v^4 + 12v$

$\phi_{vv} = -30u^4 + 180u^2v^2 - 30v^4 - 12v$

$\Rightarrow \nabla^2\phi = 0$ in w-plane.

(b) Must show $\phi = \sin(u^3 - 3uv^2) \times \cos(3u^2v - v^3)$

satisfies $\nabla^2\phi = 0$: tedious and boring, but can do.