Question

Two-dimensional fluid flow takes place in the first quadrant of the (x, y)-plane. The stream function for the flow is given by

$$\psi(x, y) = Cxy$$

where C is a positive constant.

(i) Determine the flow velocity components.

(ii) Show that the flow is irrotational and incompressible.

(iii) Sketch the streamlines of the flow

The stream function ψ is now regarded as the outer flow of a high Reynolds number steady viscous flow (with no body forces) and we wish to examine the boundary later near the wall $y = 0$. Derive the (dimensional) boundary layer equations

$$\begin{aligned} uu_x + vu_y &= C^2 x + \nu u_{yy} \\ u_x + v_y &= 0 \end{aligned}$$

where u and v are the velocity components of the flow, and give suitable boundary conditions for these equations.
Verify that a similarity solution exists in the form

$$\psi = xf(y)$$

and determine the differential equation satisfied by f, giving suitable boundary conditions.

Answer

We have $\phi = Cxy$, flow in $x \geq 0$, $y \geq 0$.

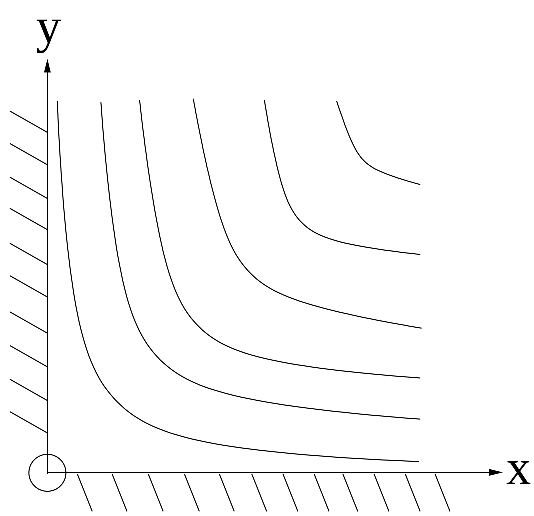
(i) $\left.\begin{array}{l} u = \phi_y = Cx \\ v = -\phi_x = -Cy \end{array}\right\} \Rightarrow \underline{q} = (Cx, -Cy, 0)$

(ii) $curl\underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Cx & -Cy & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ irrotational.

Also $div(\underline{q}) = \dfrac{\partial}{\partial x}(Cx) + \dfrac{\partial}{\partial y}(-Cy) + \dfrac{\partial}{\partial z}(0) = C - C = 0.$

(iii) $\phi = 0$ on $x = 0$, $y = 0$

$\phi = constant = \beta$ say $\Rightarrow xy = constant \Rightarrow$hyperbolae.



Now consider the Navier-Stokes equations

$$\left.\begin{array}{c} \underline{q}_t + (\underline{q}.\nabla)\underline{q} = \dfrac{-1}{\rho}\nabla p + \nu\nabla^2\underline{q} \\ div(\underline{q}) = 0 \end{array}\right\} \begin{array}{l} \text{Non} - \text{dimensionalise with} \\ \underline{x} = L\overline{\underline{x}}, \quad \underline{q} = U\overline{\underline{q}}, \quad p = \rho U^2\overline{p} \end{array}$$

Where L and U are a representitive length and speed.

(Dropping bars)

$$\frac{U^2}{L}(\underline{q}.\nabla)\underline{q} = \frac{-U^2}{L}\nabla p + \frac{\nu U}{L^2}\nabla^2\underline{q} = 0$$

$$div(\underline{q}) = 0$$

Thus the momentum equation becomes

$$(\underline{q}.\nabla)\underline{q} = -\nabla p + \frac{1}{Re}\nabla^2 \underline{q}, \quad (Re = \frac{LU}{\nu})$$

Away from the boundaries in the flow, since $Re \gg 1$ the flow is essentially inviscid so that $\phi = Cxy$. But near $y = 0$ we must rescale $y = \delta\tilde{y}$, $v = \delta\tilde{v}$. $(\delta \ll 1)$

$\Rightarrow$

$$\begin{aligned} uu_x + \tilde{v}u_{\tilde{y}} &= -p_x + \frac{1}{Re}\left(u_{xx} + \frac{1}{\delta^2}u_{\tilde{y}\tilde{y}} \right) \\ \delta(u\tilde{v}_x + \tilde{v}\tilde{v}_{\tilde{y}}) &= \frac{1}{Re}\left(\delta\tilde{v}_{xx} + \frac{1}{\delta}\tilde{v}_{\tilde{y}\tilde{y}} \right) \\ u_x + \tilde{v}_{\tilde{y}} &= 0 \end{aligned}$$

Now consider the size of δ.

If $\delta^2 Re \ll 1$ then $u_{\tilde{y}\tilde{y}} = 0$ to leading order, and this can never match with the outer flow. If $\delta^2 Re \gg 1$ then we just get back to the inviscid equations.

$\Rightarrow \delta^2 Re = 1$, so $\delta = \frac{1}{\sqrt{Re}}$.

Then the leading order equations (redimensionalized) are

$$\left.\begin{aligned} uu_x + vu + y = &\ \tfrac{-1}{\rho}p_x + \nu u_{yy} \\ 0 = &\ p_y \\ u_x + v_y = &\ 0 \end{aligned}\right\}$$

Outer flow: $p + \frac{1}{2}p\underline{q}^2 = constant, \quad \Rightarrow p_x = -\rho u u_x$

But $u = Cx, \quad \Rightarrow -p_x = \rho C^2 x$

$$\begin{aligned} \Rightarrow uu_x + vu_y &= C^2 x + \nu u_y y \\ u_x + v_y &= 0 \end{aligned}$$

Boundary conditions, $u = v = 0$ on $y = 0$ (no slip), $u \to Cx$ as $y \to \infty$ (matching).

Now with $\phi = xf(y)$ $\quad \begin{matrix} u = xf' & u_y = xf'' & u_{yy} = xf''' \\ v = -f & u_x = f' & \end{matrix}$

$\Rightarrow xf'^2 - fxf'' = C^2 x + \nu x f'''$

$\Rightarrow f'^2 - ff'' - C^2 - \nu f''' = 0$

$\Rightarrow \nu f''' + ff'' - f'^2 + C^2 = 0$

Boundary conditions: $f(0) = f'(0) = 0, \quad f'(\infty) = c$