

Question

A viscous fluid of constant density ρ and constant kinematic viscosity ν flows steadily in a two-dimensional channel $-\infty < x < \infty$, $0 \leq z \leq h$. There are no body forces. Two solid planes at $z = 0$ and $z = h$ confine the fluid and flow is produced by a pressure gradient $p_x = -A$ where A is a constant. By assuming a velocity of the form $\underline{q} = (u(z), 0, 0)$ show that the velocity is given by

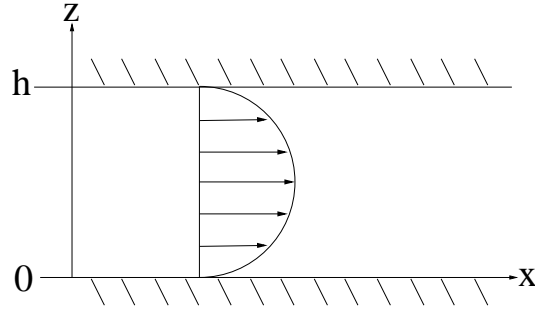
$$u = -\frac{A}{2\nu}(z^2 - zh)$$

and find the mass flow per unit width of the channel.

The EZflow ice cream company makes fancy ice cream products in a large production room at the heart of the EZfood Corporation factory. Ice cream is fed to the production room by two different channels, both of which may be considered to be two-dimensional channels of depth h . Ashok makes products from the output of channel A , whilst Betty makes products from the output of channel B . The ice cream in channel A has constant density ρ_A and constant kinematic viscosity ν_A and is driven by a constant pressure gradient $-A$.

One wall of channel B lies against a heat exchanger, so that though the ice cream in channel B has constant density ρ_B , its kinematic viscosity is given by $\nu_B e^{-\alpha z}$ where ν_B and α are constants and $0 \leq z \leq h$. By again assuming a solution of the form $\underline{q} = (u(z), 0, 0)$ find an exact solution to the Navier-Stokes equations in channel B . Use your exact solution to determine the (constant) pressure gradient $-B$ that must be imposed in channel B if Ashok and Betty are to be supplied with the same amount of ice cream, thereby avoiding a damaging industrial dispute.

Answer



Seek $\underline{q} = (u(y), 0, 0)^T$. Then certainly $\text{div}(\underline{q}) = 0$ OK.

Now we have steady flow, $\underline{q}_t = 0$. $\underline{q} = (u(y), 0, 0)^T \Rightarrow (\underline{q} \cdot \nabla)\underline{q} = 0$.

Also $p_x = -A$. So Navier-Stokes \Rightarrow

$$0 = A/\rho + \nu(u_{xx} + u_{yy} + u_{zz}) = A/\rho + \nu u_{zz}$$

$$\left. \begin{aligned} 0 &= -py/\rho + \nu(0) \\ 0 &= -px/\rho + \nu(0) \end{aligned} \right\} p = p(x) \text{ only}$$

$$\Rightarrow \frac{-A}{\mu} = u_{zz} \Rightarrow u_z = \frac{-Az}{\mu} + K_1, u = \frac{-Az^2}{2\mu} + K_1z + K_2$$

$$\text{No slip} \Rightarrow u(0) = u(h) = 0 \Rightarrow K_2 = 0, 0 = \frac{-Ah^2}{2\mu} + K_1h$$

$$\Rightarrow u = \frac{-A}{2\mu}(z^2 - zh)$$

Volume flow in channel $= \rho \int_0^h u dz$.

$$= \frac{-A\rho}{2\mu} \int_0^h (z^2 - zh) dz = \frac{-A\rho}{2\rho} \left[\frac{z^3}{3} - \frac{z^2h}{2} \right]_0^h = \frac{A\rho h^3}{2\mu b} = \frac{A\rho h^3}{12\mu}$$

Now consider the Ice-cream factory:-

By the above, Ashok's tube has mass flow

$$M_A = \frac{Ap_A h^3}{12\mu_A}$$

Now in channel B, we have, with $\underline{q} = (u(z), 0, 0)^T$ and $p_A = -B$ say

$$0 = \frac{B}{\rho_B} + \nu_B e^{-\alpha z} u_{zz}$$

and of course $\text{div}(\underline{q})$

$$\Rightarrow \frac{-B}{\rho_B \nu_B} e^{\alpha z} = u_{zz}$$

$$u_z = \frac{-B e^{\alpha z}}{\alpha \rho_B \nu_B} + K_1, u = \frac{-B e^{\alpha z}}{\alpha^2 \rho_B \nu_B}$$

$$u(0) = 0 \Rightarrow 0 = \frac{-B}{\alpha^2 \rho_B \nu_B} + K_2,$$

$$K_2 = \frac{B}{\alpha^2 \rho_B \nu_B}$$

$$u(h) = 0 \Rightarrow 0 = \frac{-B e^{\alpha h}}{\alpha^2 \rho_B \nu_B} + K_1 h + \frac{B}{\alpha^2 \rho_B \nu_B}, K_1 = \frac{-B(1 - e^{\alpha h})}{\alpha^2 \rho_B \nu_B h}$$

$$\begin{aligned}
u &= \frac{-Be^{\alpha z}h}{\alpha^2\rho_B\nu_Bh} - \frac{Bz(1 - e^{\alpha h})}{\alpha^2\rho_B\nu_Bh} + \frac{Bh}{\alpha^2\rho_B\nu_Bh} \\
\Rightarrow u &= \frac{B}{\alpha^2\rho_B\nu_Bh}(h - z - he^{\alpha z} + ze^{\alpha h}) \\
\text{So } M_B &= \rho_B \int_0^h u dz = \frac{B}{\alpha^2\rho_B\nu_Bh} \left(h^2 - \frac{h^2}{2} - \frac{h}{\alpha}(e^{\alpha h} - 1) + \frac{h^2}{2}e^{\alpha h} \right)
\end{aligned}$$

$$\begin{aligned}
M_B &= \frac{B\rho_B}{\alpha^2\mu_Bh} \left(\frac{h^2}{2}(1 + e^{\alpha h}) - \frac{h}{\alpha}(e^{\alpha h} - 1) \right) \\
&= \frac{B\rho_B}{\alpha^2\mu_B} \left(\frac{h}{2}(1 + e^{\alpha h}) - \frac{1}{\alpha}(e^{\alpha h} - 1) \right)
\end{aligned}$$

$$\begin{aligned}
\text{So } M_B = M_A \text{ when } &\frac{B\rho_B}{\alpha^2\mu_B} \left(\frac{h}{2}(1 + e^{\alpha h}) - \frac{1}{\alpha}(e^{\alpha h} - 1) \right) = \frac{A\rho_Ah^3}{12\mu_A} \\
B &= \frac{A\rho_Ah^3\alpha^2\mu_B}{\rho_B12\mu_A} \frac{1}{\left(\frac{h}{2}(1 + e^{\alpha h}) - \frac{1}{\alpha}(e^{\alpha h} - 1) \right)}
\end{aligned}$$