Question

(a) Give a brief explanation of the difference between *Eulerian* and *Lagrangian* descriptions of a fluid flow. Denoting the Eulerian coordinates as usual by $\underline{x}$, prove that for any suitably smooth function $\phi(\underline{x}, t)$

$$\frac{d\phi}{dt} = \phi_t + (\underline{q}.\nabla)\phi$$

where $\underline{q}$ is the flow velocity and d/dt denotes the Lagrangian derivative.

YOU MAY ASSUME that the equation of motion of a fluid with constant density ρ, stress tensor T and acceleration $\underline{a} = d\underline{q}/dt$ is

$$\text{div}T + \rho\underline{b} = \rho\underline{a} \longrightarrow (1)$$

where $\underline{b}$ is the body force per unit mass.

(i) State from which of Newton's laws (1) is derived and identify the physical quantity that is being conserved.

(ii) Name the physical principle that may be used to show that $T = T^T$, and therefore that the stress temsor is symmetric.

(iii) Assuming that the stress tensor for an incompressible linear viscous fluid is given by

$$T_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

derive the Navier-Stokes equations for the flow of a viscous fluid.

(b) Using tensorial notation or otherwise, prove, for suitably smooth vectors $\underline{u}$ and $\underline{v}$, the vector identities

(i)

$$\nabla \times (\underline{v} \times \underline{v}) = (\underline{v}.\nabla)\underline{u} - (\underline{u}.\nabla)\underline{v} + \underline{u}\nabla.\underline{v} - \underline{v}\nabla.\underline{u}$$

(ii)

$$\nabla \times (\nabla \times \underline{u}) = \nabla(\nabla.\underline{u}) = \nabla^2\underline{u}$$

Answer

EULERIAN $\frac{\partial}{\partial x}$, fix attention on point $\underline{x}$ fixed in space; seek to determine velocity $\underline{q}(\underline{x}, t)$

LAGRANGIAN $\frac{\partial}{\partial X}$, fix attention on and follow a given fluid particle with position $\underline{X}$ at $t = 0$; seek to determine the motion via $\underline{x} = \underline{x}(\underline{X}, t)$

Now $\left.\dfrac{d\phi}{dt}\right|_{\underline{X}}$ means fix $\underline{X}$. Relative to $\underline{x}$ instead, use chain rule.

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t}\frac{dt}{dt} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

But by definition $\frac{\partial x}{\partial t} = u$, $\frac{\partial y}{\partial t} = v$, $\frac{\partial z}{\partial t} = w$.
$\Rightarrow \left.\dfrac{d\phi}{dt}\right|_{\underline{X}} = \dfrac{\partial \phi}{\partial t} + u\phi_x + v\phi_y + w\phi_t = \phi_t + (\underline{q}.\nabla)\phi$
Now we have $div(T) + \rho \underline{b} = \rho \underline{\alpha}$

(i) This is derived from Newtons 2nd law; linear momentum is being conserved.

(ii) Conservation of angular (moment of) momentum leads to the symmetry of the stress tensor.

(iii) With

$$T_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

$$\begin{aligned} div(T) = \frac{\partial T_{ij}}{\partial x_j} &= -\delta_{ij}\frac{\partial p}{\partial x_j} + \mu \frac{\partial}{\partial x_j}(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}) \\ &= -\frac{\partial p}{\partial x_i} + \mu \nabla^2 \underline{q} + \mu \nabla(\nabla.\underline{q}) \end{aligned}$$

So $divT = -\nabla p + \mu \nabla^2 \underline{q} + \mu \nabla(\nabla.\underline{q})$

Now $\nabla.\underline{q} = 0$ since incompressible.
$\Rightarrow -\nabla p + \mu \nabla^2 q + \rho \underline{b} = \rho \underline{a} = \rho(\underline{q}_t + (\underline{q}.\nabla)\underline{q})$
$\Rightarrow \underline{q}_t + (\underline{q}.\nabla)\underline{q} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{q}$ (Navier-Stokes)

(b)

$$
\begin{aligned}
\nabla \times (\underline{u} \times \underline{v}) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\underline{u} \times \underline{v})_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} u_l v_m) \\
&= \epsilon_{kij} \epsilon_{klm} (\frac{\partial u_l}{\partial x_j} v_m + u_l \frac{\partial v_m}{\partial x_j}) \\
&= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{mi})(v_m \frac{\partial u_l}{\partial x_j} + u_l \frac{\partial v_m}{\partial x_j}) \\
&= v_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial v_j}{\partial x_j} - v_i \frac{\partial u_j}{\partial x_j} - u_j \frac{\partial v_m}{\partial x_j} \\
&= (\underline{v}.\nabla)\underline{u} + \underline{u} div(\underline{v}) - \underline{v} div(\underline{u}) - (\underline{u}.\nabla)\underline{v}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (\nabla \times \underline{u}) &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \underline{u})_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} \frac{\partial u_m}{\partial x_l}) \\
&= \epsilon_{kij} \epsilon_{klm} \frac{\partial^2 u_m}{\partial x_j \partial x_l} = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
&= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nabla(\nabla.\underline{u}) - \nabla^2 \underline{u}
\end{aligned}
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