QUESTION

Are the following true or false? Give a proof or a counterexample, as appropriate.

(i) if $\gcd(a, b) = d$, then $\gcd(a + d, b) = d$.

(ii) If $\gcd(a, b) = 1$ and $c|a$, then $\gcd(c, b) = 1$.

(iii) If $\gcd(a, b)$=$\gcd(a, c) = d$, then $\gcd(b, c) = d$.

(iv) If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.

(Hint: Corollary 1.5 will help here.)

ANSWER

(i) FALSE: e.g. gcd(10,12)=2, but gcd(10+2,12)=gcde(12,12)=12$\neq$ 2.

(ii) TRUE: Let $\gcd(c, b) = d$, then $d|b$ and $d|c$, and so, since $c|a$, we have $d|a$ (th.1.3(3)). Thus d is a common divisor of a and b, so $d \leq$$\gcd(a, b) = 1$. Since, by definition of gcd, we already know that $d \geq 1$, the result follows.

(iii) FALSE: e.g. gcd(2,4)=gcd(2,8)=2, but gcd(4,8)=4$\neq$2.

(iv) TRUE: $\gcd(a, b)$=$\gcd(a, c) = 1$, so by cor.1.5 we can find integers x, y, u, v such that $ax + by = 1$ and $au + cv = 1$. Multiplying these together and rearranging give:-

$$1 = (ax + by)(au + cv) = a(axu + xcv + uby) + bc.yv$$

Thus we have found integers $r = axu + xcv + uby$ and $s = yv$ such that $1 = ar + (bd)s$, and then cor.1.5 gives $\gcd(a, bc) = 1$ as required.