CONTINUED FRACTIONS
INVERSE PERIODS AND A THEOREM OF GALOIS

Let $\alpha_0 = [\overline{a_0, \ldots, a_{k-1}}]$ be a purely periodic continued fraction of period k. The α_n are also reduced, and satisfy

$$\alpha_0 = a_0 + \frac{1}{\alpha_1} \ \alpha_1 = a_1 + \frac{1}{\alpha_2} \ldots \ \alpha_{k-1} = a_{k-1} + \frac{1}{\alpha_0}$$

using periodicity.
Reversing the sequence, rearranging and taking conjugates gives

$$-\frac{1}{\overline{\alpha_0}} = a_{k-1} - \overline{\alpha_{k-1}}, \ -\frac{1}{\overline{\alpha_{k-1}}} = a_{k-2} - \overline{\alpha_{k-1}} \ldots \ -\frac{1}{\overline{\alpha_1}} = a_0 - \overline{\alpha_0}$$

Write $-\frac{1}{\overline{\alpha_n}} = \beta_n$. Then $\beta_n > 1$ and

$$\beta_0 = a_{k-1} + \frac{1}{\beta_{k-1}} \ \beta_{k-1} = a_{k-1} + \frac{1}{\beta_{k-1}}, \ldots, \ \beta_1 = a_0 + \frac{1}{\beta_0}$$

From which we deduce

$$|beta_0 \left(= -\frac{1}{\overline{\alpha_0}} \right) = [\overline{a_{k-1}, a_{k-2}, \ldots, a_1, a_0}]$$

We can also investigate the complete quotients, developing formulae of use later.
Let $\alpha_0 = \frac{\sqrt{D}+P_0}{Q_0}$ where $Q_0 | D - P_0^2$
then $\alpha_n = \frac{\sqrt{D}+p_n}{Q_n}$ and $Q_n | D - P_n^2$ so $\frac{\sqrt{D}+P_n}{Q_n} = a_n + \frac{Q_{n+1}}{\sqrt{D}+P_{n+1}}$
Clearing of fractions and equating rational and irrational parts gives

$$\begin{aligned} D + P_n P_{n+1} &= a_n Q_n P_{n+1} + Q_n Q_{n+1} \\ P_n + P_{n+1} &= a_n Q_n \end{aligned}$$

Multiplying the second equation by P_{n+1} and subtracting gives

$$D - P_{n+1}^2 = Q_n Q_{n+1}$$

which we have met before.
Now $\alpha_n = \frac{\sqrt{D}+P_n}{Q_n}$ so $\overline{\alpha_n} = \frac{-\sqrt{D}+P_n}{Q_n}$
Thus

$$\beta_n = -\frac{1}{\overline{\alpha_n}} = \frac{Q_n}{\sqrt{D} - P_n} = \frac{Q_n(\sqrt{D} + P_n)}{D - P_n^2} = \frac{\sqrt{D} - P_n}{Q_{n-1}}$$

This needs interpreting for $n = 0$.
However

$$\beta_0 = \beta_k \text{ by periodicity } = \frac{\sqrt{D} + P_k}{Q_{k-1}} = \frac{\sqrt{D} + P_0}{Q_{k-1}}$$

This relates the complete quotients of α_0 and $-\frac{1}{\overline{\alpha_0}}$
Finally we deduce
Theorem (Serret)
Two conjugate quadratic irrationals have inverse periods (not necessarily reduced).
Proof
Let $\alpha_0 = [a_0, \dots a_{k-1}\overline{a_k, \dots a_{m+k-1}}]$

$$\alpha_0 = \frac{p_{k-1}\alpha_k + p_{k-2}}{q_{k-1}\alpha_k + q_{k-1}},$$

α_k purely periodic.
so $-\frac{1}{\overline{\alpha_k}} = [\overline{a_{m+k-1}, \dots a_k}]$. However

$$\begin{aligned} \overline{\alpha_0} &= \frac{p_{k-1}\overline{\alpha_k} + p_{k-1}}{q_{k-1}\overline{\alpha_k} + q_{k-2}} \\ &= \frac{p_{k-2}\left(-\frac{1}{\overline{\alpha_k}}\right) + (-p_{k-1})}{q_{k-2}\left(-\frac{1}{\overline{\alpha_k}}\right) + (q_{k-1})} \end{aligned}$$

So $\overline{\alpha_0}$ and $-\frac{1}{\overline{\alpha_k}}$ are equivalent and so their continued fractions agree from some point on. Thus $\overline{\alpha_0}$ has the reverse period of α_0
Examples

$$\begin{aligned} \frac{14 - \sqrt{37}}{3} &= [2, 1, 1, \overline{1, 3, 2}] \\ \frac{14 + \sqrt{37}}{3} &= [6, \overline{1, 2, 3}] = [6; 1, \overline{2, 3, 1}] \end{aligned}$$

If α_0 and $\overline{\alpha_0}$ are equivalent then they agree from some point onward. This means that the have the same period. They also have inverse periods, but this does not mean that the period is symmetric, because of the shift noted above.
Examples
$\alpha_0 = \frac{\sqrt{7}+3}{2}$ $\overline{\alpha_0} = \frac{\sqrt{7}-3}{-2}$

α_0	k	0	1	2	3	4	5
	P_k	3	12211				
	Q_k	2	3	1	3	2	3
	a_k	3	1	4	1	1	

$\alpha_0 = [3; \overline{1,4,1,1}]$

$\alpha_0 - \overline{\alpha_0} = 3$ so $\overline{\alpha_0}$ and α_0 are equivalent. $\left(\overline{\alpha_0} = \frac{1.\alpha_0+3}{0.\alpha_0+1}\right)$

α_0	k	0	1	2	3	4	5	6
	P_k	-3	3	2	1	1	2	2
	Q_k	-2	1	3	2	3	1	3
	a_k	0	5	1	1	1	4	1

$$\begin{aligned} \overline{\alpha_0} &= [0, 5, \overline{1,1,1,4}] \\ &= [0; 5, 1, \overline{1,1,4,1}] \text{ - reverse period of } \alpha \\ &= [o; 5, 1, 1, \overline{1,4,1,1}] \text{ - same period as } \alpha \end{aligned}$$

because this shift starts somewhere, the period can be slit into 2 symmetric parts. In this case 1 and 1,4,1.

Square roots of rationals

Let $d \in Q$, not the square of a rational, and $d > 1$. Then $-\sqrt{d} < -1$ and the continued function for $\sqrt{D}$ has one term before the period.

$\sqrt{d} = [a_0; \overline{a_1, \ldots, a_k}]$ so $\frac{1}{\sqrt{d}-a_0} = [\overline{a_1, \ldots, a_k}] = \alpha_0$

$-\frac{1}{\overline{\alpha_0}} = \sqrt{d} + a_0 = [\overline{a_k, \ldots, a_1}]$

but $\sqrt{d} + a_0 = [2a_0, \overline{a_1, \ldots, a_k}]$

Thus by uniqueness

$$a_k = 2a_0, \;\; a_{k-1} = a_1, \ldots$$

so $\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$

(If $k = 1$ the symmetric part is empty)

Conversely we argue as follows

Suppose $\alpha_0 = [a_o; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$. $2a_0 \neq 0$ so $\alpha_0 > 1$

$\frac{1}{\alpha - -a_0} = [\overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$ and so $a_0 - \overline{\alpha_0} = [\overline{2a_0, a_1, a_2, \ldots, a_1}]$ so $-\overline{\alpha_0} = [a_0, \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}]$

Thus $\alpha_0 = -\overline{\alpha_0}$ and so α_0 is the square root of a rational number.