CONTINUED FRACTIONS
BEST APPROXIMATIONS

$\frac{a}{b}$ is said to be a best approximation to α ($\alpha \in Z$ $b \in N$) if

$$\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{a}{b}\right| \Rightarrow q > b.$$

We now prove that the convergents to an (irrational) number give a sequence of best approximations.
Note that as in the previous result, we often investigate $|q\alpha - p|$ rather than $\left|\alpha - \frac{p}{q}\right|$. Inequalities involving the former are often a bit stronger the those involving the latter.
Theorem
If $|q\alpha - p| < |q_n\alpha - p_n|$, $n > 0$ where $\frac{p_n}{q_n}$ is a convergent of the continued fraction for α, then $q > q_n$.
Proof
Assume that $|q\alpha - p| < q_n\alpha - p_n|$ and that $q \leq q_n$. It follows that $q < q_{n+1}$ $(n > 0)$. Consider the equations

$$x.p_n + y.p_{n+1} = p$$
$$x.q_n + y.q_{n+1} = q$$

$p_nq_{n+1} - p_{n+1}q_n = (-1)^n$, so this pair of equations has integer solutions x, y.
Now $y = 0 \Rightarrow p = xp_n$ $q = xq_n$, $x \neq 0$ and so $|q\alpha - p| = |x||q_n\alpha - p_n| \geq |q_n\alpha - p_n|$
If $x = 0$ then $y \neq 0$ and $q = yq_n$ which contradicts $q \leq q_n$.
So x and y are non-zero.
We now show that x and y are of opposite sign.

$$0 < q = xq_n + yq_{n+1} < q_{n+1}$$

x and y cant both be < 0 as $q > 0$
x and y cant both be > 0 otherwise $> q_{n+1}$
Now $q_n\alpha - p_n$ and $q_{n+1}|alpha - p_{n+1}$ have opposite signs, since the convergents alternate either side of α, so $x(q_n\alpha - p_n)$ and $y(q_{n+1}\alpha - p_{n+1})$ have the same sign.
Also

$$q\alpha - p = x(q_n\alpha - p_n) + y(q_{n+1}\alpha - p_{n+1})$$

so

$$
\begin{aligned}
|q\alpha - p| &= |x(q_n\alpha - p_n)| + |y(q_{n+1}\alpha - p_{n+1})| \\
&> |x(q_n\alpha - p_n)| \geq |q_n\alpha - p_n|
\end{aligned}
$$

This contradiction proves the theorem.
This proof goes back to Legendre, and is quoted in Perron.
Now $\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$
and$q \leq q_n$ multiplying the inequalities

$$\Rightarrow |q\alpha - p| < |q_n\alpha - p_n| \Rightarrow q > q_n$$

So the convergents are the best approximations to α. But how good are they?
Now we have already seen the equation

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\alpha_{n+1}q_n + q_{n-1})}$$

so

$$\left|alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} \leq \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_nq_{n+1}} < \frac{1}{q_n^2}$$

We already know from Dirichlet's theorem that an irrati0onal α has infinitely many rational approximations $\frac{p}{q}$ satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.
The sequence of convergents supplies such a set.
This does not give them all however. E.g. consider rational approximations to $\frac{779}{207}$
The convergents are

$$3,\ 4,\ \frac{15}{4},\ \frac{64}{17},\ \frac{143}{38},\ \frac{779}{207}.$$

$$
\begin{aligned}
\frac{779}{207} - \frac{79}{21} &= \frac{6}{4347} \approx 1.38 \times 10^{-3} \\
\frac{1}{21^2} &\approx 2.27 \times 10^{-3}
\end{aligned}
$$

However, notice that $\frac{79}{21} = \frac{15+64}{4+17}$
I shall not pursue this, but instead show that if $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ $(p, q) = 1$ then $\frac{p}{q}$ is one of the convergents of the continued fraction for α.
Proof

Suppose not. Then $q_n \leq q \leq q_{n+1}$ determines an integer n, and $|q\alpha - \phi| < |q_n\alpha - p_n|$ is impossible. (The earlier theorem can be improved to $q \geq q_{n+1}$) so $|q_n\alpha - p_n| \leq |q\alpha - p| < \frac{1}{2q}$
i.e $\left|\alpha - \frac{p_n}{q_n}\right| Z < \frac{1}{2qq_n}$
Now

$$\begin{aligned}
\frac{1}{qq_n} &\leq \frac{|qp_n - pq_n|}{qq_n} \text{ (even if } q = q_n \ \frac{p}{q} \neq \frac{p_n}{q_n}) \\
&= \left|\frac{p_n}{q_n} - \frac{p}{q}\right| \\
&\leq \left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p}{q}\right| \\
&< \frac{1}{2qq_n} + \frac{1}{2q^2} \\
\frac{1}{2qq_n} &< \frac{1}{2q^2}
\end{aligned}$$

so $q < q_n$ This is a contradiction so the theorem is proved.
Now of any two successive convergents, at least one satisfies $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$
Proof
Since the convergents are alternatively greater and less then x

$$\left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right| + \left|\alpha - \frac{po_n}{q_n}\right|$$

Suppose the result false. Then

$$\begin{aligned}
\frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2} &\leq \left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right| + \left|\alpha - \frac{p_n}{q_n}\right| \\
&= \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| \\
&= \frac{1}{q_nq_{n+1}}
\end{aligned}$$

i.e. $\left(\frac{1}{q_{n+1}} - \frac{1}{q_n}\right)^2 \leq 0$ i.e. $q_{n+1} = q_n$. This is true only if $n = 1$ $a_1 = 1$ $q_1 = q_0 = 1$. Otherwise $q_{n+1} > q_n$.
Even in this case

$$0 < \frac{p_1}{q_1} - x = 1 - \frac{1}{1+}\frac{1}{a_2+} < 1 - \frac{a_2}{a_2+1} \leq \frac{1}{2}$$

so the theorem is still true.
Further, of any three successive convergents, at least one satisfies $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{5}}$
Proof

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} = \frac{1}{q_n^2\left(\alpha_{n+1} + \frac{q_{n-1}}{q_n}\right)}$$

Now suppose that

$$\alpha_i + \frac{q_{i-2}}{q_{i-1}} \leq \sqrt{5} \text{ for } i = n-1, n, n+1$$

Then

$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n} \text{ and } \frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \frac{q_{n-3}}{q_{n-2}}$$

so

$$\frac{1}{\alpha_n} + \frac{q_{n-1}}{q_{n-2}} = \alpha_{n-1} + \frac{q_{n-3}}{q_{n-2}} \leq \sqrt{5}$$

by assumption and

$$\begin{aligned} 1 &= \alpha_n \frac{1}{\alpha_n} \leq \left(\sqrt{5} + \frac{q_{n-2}}{q_{n-1}}\right)\left(\sqrt{5} - \frac{q_{n-1}}{q_{n-2}}\right) \\ &= 5 + 1 - \sqrt{5}\left(\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-1}}{q_{n-2}}\right) \end{aligned}$$

giving $\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-1}}{q_{n-2}} \leq \sqrt{5}$
In fact since LHS is rational we have strictly less then, so

$$\begin{aligned} \left(\frac{q_{n-1}}{q_{n-1}}\right)^2 - \left(\frac{q_{n-2}}{q_{n-1}}\right)\sqrt{5} + 1 &< 0 \\ \left(\frac{q_{n-2}}{q_{n-1}} - \frac{1}{2}\sqrt{5}\right)^2 &< \frac{1}{4} \text{ i.e.} \\ \frac{q_{n-2}}{q_{n-1}} &> \frac{1}{2}(\sqrt{5} - 1) \end{aligned}$$

This has used $i = n-1, n$. Using $1 = n, n+1$ gives

$$\frac{q_{n-1}}{q_n} > \frac{1}{2}(\sqrt{5} - 1)$$

Now $q_n = a_n q_{n-1} + q_{n-2}$

$$a_n = \frac{q_n}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}} < \frac{2}{\sqrt{5}-1} - \frac{1}{2}(\sqrt{5}-1) = 1$$

$a_n < 1$ is a contradiction.
Now let $\alpha = \frac{1}{2}(\sqrt{5}-1) = [0, 1, 1, 1, \ldots]$
Suppose that there are an infinite number of solutions of

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Aq^2} \;\; A > \sqrt{5}$$

$\alpha = \frac{p}{q} + \frac{\delta}{q^2}$ where $|\delta| < \frac{1}{A} < \frac{1}{\sqrt{5}}$
Then $\frac{\delta}{q} = q\alpha - p$ and

$$\frac{\delta}{q} - \frac{1}{2}q\sqrt{5} = q\left(\frac{1}{2}\sqrt{5} - 1\right) - p - \frac{1}{2}q\sqrt{5} = -\frac{1}{2}q - p$$

so

$$\left(\frac{\delta}{q}\right)^2 - \delta\sqrt{5} + \frac{5}{4}q^2 = \left(\frac{1}{2}q + p\right)^2$$

so

$$\left(\frac{\delta}{q}\right)^2 - \delta\sqrt{5} = p^2 pq - q^2$$

when q is large, since $|\delta|\sqrt{5} < 1$ the LHS is between -1 and $+1$ whereas RHS is an integer.
So $p^2 + pq - q^2 = 0$ i.e.$(2p+q)^2 = 5q^2$ which is impossible for integers p and q. So $\frac{1}{\sqrt{5}}$ is the best possible. This establishes Hurwitz theorem.
We now investigate for which numbers $\sqrt{5}$ is best possible. It turns out that the criterion is that these numbers should end in an infinite tail of 1's. We generalise this.
Definition
Two irrational numbers α and β are equivalent if they have the same tail to their continued fraction, in the sense that
$\alpha = [a_0; a_1, \ldots, a_k, c_0, c_1 c_2 \ldots]$
$\beta = [b_0; b_1, \ldots, b_j, c_0, c_1, c_2 \ldots]$
Theorem
Two irrational numbers α and β are equivalent if and only if there exist integers a, b, c, d with $ad - bc = \pm 1$ such that

$$\alpha = \frac{A\beta + B}{C\beta + D}$$

Lemma
if $x = \frac{P\xi+R}{Q\xi+S}$ where $\xi > 1$, $PS - RQ = \pm 1$, and $Q > S > 0$ then $\frac{r}{S}$ and $\frac{P}{Q}$ are two consecutive convergents to the continued function for x. If $\frac{R}{S}$ is the $(n-1)$th, $\frac{P}{Q}$ is the nth and ξ is the $(n-1)$th complete quotient.
Proof

$$\frac{P}{Q} = [a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}$$

n can be even or odd. Choose it so that $PS - QR = (-1)^{n-1}$
$(p, q) = 1$ so $P = p_n$ $Q = q_n$
so $p_n S - q_n R = (-1)^{n-1} = p_n q_{n-1} - p_{n-1} q_n$
so $p_n(S - q_{n-1}) = q_n(R - p_{n-1})$
so $q_n | S - q_{n-1}$ since $(p_n, q_n) = 1$.
Now

$$\begin{aligned} q_n &= Q > S > 0 \\ q_n &\geq q_{n-1} > 0 \text{ so} \\ q_n &> |S - q_{n-1}| \end{aligned}$$

Hence $S - q_{n-1} = 0$ and so $R - p_{n-1} = 0$ thus
$\frac{R}{S} = \frac{p_{n-1}}{q_{n-1}}$ and $x = \frac{p_n\xi + p_{n-1}}{q_n\xi + q_{n-1}}$
i.e.

$$\begin{aligned} x &= [a_0, a_1, \ldots, a_n, \xi] \\ &= [a_0, a_1, \ldots a_n, c_0, c_1] \end{aligned}$$

where$\xi = [c_0; c_1, c_2 \ldots]$ and $c_0 \neq 0$ as $\xi > 1$ and so ξ is the $n + 1$th complete quotient.
Proof of theorem
Suppose $\alpha = [a_0, \ldots a_k, c_0, c_1 \ldots] = [a_0, \ldots a_k, w]$
$\beta = [b_0, \ldots b_j, c_o, c_1 \ldots] = [b_0, \ldots b_j, w]$
then

$$\alpha = \frac{p_k w + p_{k-1}}{q_k w + q_{k-1}} \qquad p_k q_{k-1} - p_{k-1} q_k = \pm 1$$

$$\beta = \frac{p'_j w + p'_{j-1}}{q'_j w + q'_{j-1}} \qquad\qquad p'_j q'_{j-1} - p'_{j-1} q;_j = \pm 1$$

eliminating w will give

$$\alpha = \frac{A\beta + B}{C\beta + D} \text{ where } AD - BC = \pm 1.$$

Now suppose

$$\alpha = \frac{A\beta + B}{C\beta + D} \; AD - BC = \pm 1$$

assume w.l.o.g. $C\beta + D > 0$.
Let $\beta = [b_0, \ldots b_{k-1}\beta_k] = \frac{p_{k-1}\beta_k + p_{k-2}}{q_{k-1}\beta_k + q_{k-2}}$
substituting fo β in $\alpha = \frac{A\beta+B}{C\beta+D}$ gives

$$\alpha = \frac{P\beta_K + R}{q\beta_k + s}$$

where

$$\begin{aligned} P &= Ap_{k-1} + Bq_{k-1} \\ R &= Ap_{k-2} + Bq_{k-2} \\ Q &= Cp_{k-1} + Dq_{k-1} \\ S &= Cp_{k-2} + Dq_{k-2} \end{aligned}$$

So $P, Q, R, S \in Z$ and

$$PS_Q R = (AD - BC)(p_{k-1}q_{k-2} - p_{k-1}q_{k-1}) = \pm 1$$

Now $\left|\beta - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{q_{k-1}^2}$ and $\left|\beta - \frac{p_{k-2}}{q_{k-2}}\right| < \frac{1}{q_{k-2}^2}$
so

$$p_{k-1} = q_{k-1}\beta + \frac{\varepsilon}{q_{k-1}}; p_{k-1} = q_{k-1}\beta + \frac{\varepsilon'}{q_{k-1}}$$

where $|\varepsilon| < 1$ and $|\varepsilon'| < 1$.
So

$$\begin{aligned} Q &= (C\beta + D)q_{k-1} + \frac{C\varepsilon}{q_{k-1}} \\ S &= (C\beta + D)q_{k-2} + \frac{C\varepsilon'}{q_{k-2}} \end{aligned}$$

Now $C\beta + D > 0$ and $q_{k-1} > q_{k-1}$ also $q_n \to \infty$ as $n \to \infty$. So provided k is sufficiently large, $Q > S > 0$
For such k, $\alpha = \frac{P\beta_k + R}{Q\beta_k + S}$ $PS - QR = \pm 1, Q > S > 0$
so β_k is a complete quotient in the continued fraction for α by the lemma thus $\alpha = [a_0; a_1, \ldots a_m, b_k, b_{k+1} \ldots]$ i.e. α is equivalent to β.
We now define the Markov constant of an irrational number α by

$$M(\alpha) = sup \left\{ \lambda : \left| \alpha - \frac{p}{q} \right| < \frac{1}{\lambda q^2} \text{ has infinitely many solutions } \frac{p}{q} \right\}$$

So Huzwitz theorem says
$\forall \alpha \ M(\alpha \geq \sqrt{5}$ and $M\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.
We now extend this :
Theorem
If α is equivalent to β then $M(\alpha) = M(\beta)$. If α is not equivalent to $\frac{1+\sqrt{5}}{2}$ then $M(\alpha) \geq \sqrt{8}$. If α is equivalent to $1 + \sqrt{2}$ then $M(\alpha) = \sqrt{8}$.
Proof
Recall that

$$\begin{aligned} \left| \alpha - \frac{p_k}{q_k} \right| &= \frac{1}{q_k(q_k \alpha_{k+1} + q_{k-1})} \\ &= \frac{1}{q_k^2 \left(\alpha_{k+1} + \frac{q_{k-1}}{q_k} \right)} \end{aligned}$$

Thus

$$M(\alpha) = \lim_{k \to \infty} sup \left(\alpha_{k+1} + \frac{q_{k-1}}{q_k} \right)$$

Recall from the discussion of symmetric continued fractions that

$$\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots a_1]$$

so

$$\frac{q_{k-1}}{q_k} = [0; a_k a_{k-1} \ldots a_1]$$

so

$$M(\alpha) = \lim_{k \to \infty} sup \left([0; a_k, a_{k-1}, \ldots a_1] + \alpha_{k+1} \right)$$

Now if α is equivalent to β then $\beta_j = \alpha_k$ and $b_j = a_k$ for all sufficiently large k and j for which $j - k$ has a suitable fixed value h.

If the convergents of β are $\frac{P_j}{Q_j}$ then for j and k differing by h, the continued fractions for $\frac{q_{k-1}}{q_k}$ and $\frac{Q_{j-1}}{Q_j}$ have rhe same partial quotients at the beginning, and the length of agreement can be made large by making j and k sufficiently large.
Suppose $\frac{q_{k-1}}{q_k}$ and $\frac{Q_{j-1}}{Q_j}$ agree in the first l_1 partial quotients, and denote the common convergents by $\frac{r_i}{s_i}$ $(i = 0, \dots l)$
so

$$\frac{q_{k-1}}{q_k} = \frac{r_{l-1}x_l + r_{l-2}}{s_{l-1}x_l + s_{l-2}}$$

and

$$\frac{Q_{j-1}}{Q_j} = \frac{r_{l-1}y_l + r_{l-2}}{s_{l-1}y_l + s_{l-2}}$$

Then $[x_l] = [y_l] =$ common $l + 1$th partial quotient so $|x_l - y_l| \leq 1$.
Then we have

$$\left|\frac{q_{k-1}}{q_k} - \frac{Q_{j-1}}{Q_j}\right| = \frac{|x_l - y_l|}{(s_{l-1}x_l + s_{l-2})(s_{l-1}y_l + s_{l-1})} \leq \frac{1}{s_{l-1}^2}$$

Now provided j and k are large enough, we have

$$\left|\frac{q_{k-1}}{q_k} - \frac{Q_{j-1}}{Q_j}\right| < \varepsilon$$

since $s_{l-1} \geq (l-1)$th term in Fibonacci sequence.
Also for large j, k $\alpha_k = \beta_j$, so

$$\left(\alpha_k + \frac{q_{k-1}}{q_k}\right) - \left(\beta_j + \frac{Q_{j-1}}{Q_j}\right) = \frac{q_{k-1}}{q_k} - \frac{Q_{j-1}}{Q_j} \to 0 \text{ as } j, k \to \infty,\ j - k = h$$

Thus $M(\alpha) = M(\beta)$
If α is not equivalent to $\frac{\sqrt{5}+1}{2}$ then infinitely many of the a_k are ≥ 2.
If $a_k \geq 3$ for infinitely many k then

$$\begin{aligned} M(\alpha) &= \lim sup\left(\alpha_{k+1} + \frac{q_{k-1}}{q_k}\right) \\ &\geq \lim sup(a_{k+1}) \geq 3 \end{aligned}$$

So suppose that the a_k contain only 1's and 2's from some point on.
Case I

$a_k = 2$ from some point on. Then α is equivalent to $1+\sqrt{2} = [2;2,2,\ldots]$

$$\begin{aligned} M(\alpha) &= \lim sup\left(\alpha_{k+1} + \frac{q_{k-1}}{q_k}\right) \\ \alpha_{k+1} &= [2;2,\ldots] = 1+\sqrt{2} \\ \frac{q_{k-1}}{q_k} &= [0;\underbrace{2,2,\ldots}_{k \text{ times}}] \to \frac{1}{1+\sqrt{2}} \text{ as } k \to \infty \end{aligned}$$

So $M(\alpha) = 1+\sqrt{2}+\frac{1}{1+\sqrt{2}} = \sqrt{8}$
Case II
Suppose thee are infinitely many 1's and 2's.
Then there are infinitely many k such that $a_k = 1$ and $a_{k+1} = 2$, so

$$\begin{aligned} \alpha_{k+1} &= 2 + \frac{1}{a_{k+2}+}\frac{1}{a_{k+3}} \geq 2 + \frac{1}{2+\frac{1}{1}} = \frac{7}{3} \\ \frac{q_{k-1}}{q_k} &= \frac{1}{a_k+}\frac{1}{a_{k-1}+}\cdots \geq \frac{1}{1+\frac{1}{a_{k-1}}} \geq \frac{1}{1+\frac{1}{1}} = \frac{1}{2} \end{aligned}$$

So $M(\alpha) \geq \frac{7}{3}+\frac{1}{2} = \frac{17}{6} > \sqrt{8}$
Note: This shows that if $\alpha \not\sim 1+\sqrt{2}$ then $M(\alpha) \geq \frac{17}{6}$.
Theorem
There are uncountably many α with $M(\alpha) = 3$
Proof
Let $\alpha = [\underbrace{1;1,1,\ldots 1}_{r_1},2,2,\underbrace{1,1,\ldots 1}_{r_2},2,2,\underbrace{1,1,\ldots 1}_{r_3},2,2,1,\ldots]$ $r_1 < r_2 < r_3$

(i) If $a_{k+1} = 1$ then $\alpha_{k+1} < 2$ and since $\frac{q_{k-1}}{q_k} < 1$, $\alpha_{k+1} + \frac{q_{k-1}}{q_k} < 3$.

(ii) If $a_{k+1} = 2$ and $a_{k+1} = 2$ then

$$\alpha_{k+1} + \frac{q_{k-1}}{q_k} = \left(2 + \frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\cdots\right) + \left(\frac{1}{1+}\frac{1}{1+}\cdots\frac{1}{1}\right)$$

If k is large the sequences of 1's can be made as long as we like before a 2 appears. So $\alpha_{k+1} + \frac{q_{k-1}}{q_k} \to 2 + \frac{1}{2+\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2} = 3$

(iii) If $a_{k+1} = 2$ and $a_k = 1$ then

$$\alpha_{k+1} + \frac{q_{k-1}}{q_k} = \left(2 + \frac{1}{1+}\frac{1}{1+}\cdots\right) + \left(\frac{1}{2+}\frac{1}{1+}\cdots\frac{1}{1}\right) \to 2 + \frac{1}{\frac{\sqrt{5}+1}{2}} + \frac{1}{2+\frac{\sqrt{5}-1}{2}} = 3$$

So $M(\alpha) = \lim sup \left(\alpha_{k+1} + \frac{q_{k-1}}{q_k}\right) = 3$
Two such α's are equivalent iff their associated sequences of r_i's are equivalent in the same sense of having equal tails. There are uncountably many inequivalent such sequences of r_i's so uncountably many inequivalent α with $M(\alpha) = 3$.