Question

Say what is meant by a Lebesgue measurable set in R and show that every set of Lebesgue outer measure zero is measurable.

Define the Cantor ternary set, and show that it is uncountable. Show also that it has measure zero.

Give an example of a measurable function $f : [0,1] \to R$ and a Lebesgue measurable set $E \subseteq R$ for which $f^{-1}(E)$ is not a measurable set, proving your assertions.

Hint: Use for E a suitable subset of the Cantor set.

[The existence of a non-measurable subset of $[0,1]$ may be assumed.]

Answer

A set E is measurable in R iff for all $S \subset R$, $m^*(S) = m^*(S \cap E) + m^*(S - E)$ where m^* is Lebesgue outer measure. If E has outer measure zero then since $S \cap E \subseteq E$, $m^*(S \cap E) = 0$. Also $S - E \subseteq S$ and so $m^*(S - E) \leq m^*(S)$.
Further, $S = (S \cap E) \cup (S - E)$
and so $m^*(S) \leq m^*(S \cap E) + m^*(S - E) = m^*(S - E)$
Hence $m^*(S) = m^*(S - E) = m^*(S - E) + m^*(S \cap E)$.
Thus E is measurable.

The Cantor ternary K set is the set of numbers in $[0,1]$ which have a decimal expansion in scale 3 containing only 0 or 2. $m(K) = 0$.
Suppose $x \epsilon K$ and $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$ where $x_i = 0$ or 2.
Consider the mapping $f : K \to [0,1]$
$f : x \to y$ where $y = \sum_{i=1}^{\infty} \frac{\frac{1}{2}x_i}{2^i}$
The range of f is $(0,1)$, since every $y \epsilon (0,1)$ has a binary expansion. Because of ambiguity over terminating and non-terminating decimals f is not $1-1$, but each $y \epsilon (0,1)$ is the image of at most two $x \epsilon K$. It follows that K is uncountable.
Now consider the function $g : [0,1] \to K$ defined as follows. If $x \epsilon [0,1]$ then x has a unique non-terminating binary expansion

$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$, define $g(x) = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \, \epsilon K$
g is an increasing function and so is measurable, since $\{x|f(x) < c\}$ is of the form $\{x|x < a\}$ or $\{x|x \leq a\}$. Let F be a subset of $[0, 1]$ which is not measurable. Let $E = g(F) \subseteq K$, $m^*(E) = 0$ and so E is measurable. g is $1 - 1$ and so $g^{-1}(E) = F$ which is not measurable.
This therefore provides the example required.