Question
Consider the sequence $\{f_n\}$ of functions

$$f_n(x) = \begin{cases} n^2x & 0 \leq x \leq \frac{1}{n} \\ \\ 2n - n^2x & \frac{1}{n} < x \leq \frac{2}{n} \\ \\ 0 & \text{otherwise} \end{cases}$$

Evaluate $\int \lim f_n$ and $\lim \int f_n$ and comment.

Answer
$\lim\limits_{n\to\infty} f_n(x) = 0$ therefore $\displaystyle\int \lim f_n = 0$

$$\begin{aligned} \int f_n &= \int_0^{\frac{1}{n}} n^2x + \int_{\frac{1}{n}}^{\frac{2}{n}} 2n - n^2x \\ &= n^2\left[\frac{x^2}{2}\right]_0^{\frac{1}{n}} + \left[2nx - n^2\frac{x^2}{2}\right]_{\frac{1}{n}}^{\frac{2}{n}} \\ &= n^2\frac{1}{2}\frac{1}{n^2} + 2n\frac{1}{n} - \frac{n^2}{2}\frac{3}{n^2} \\ &= \frac{1}{2} + 2 - \frac{3}{2} = 1 \end{aligned}$$

Therefore $\lim f_n = 1$