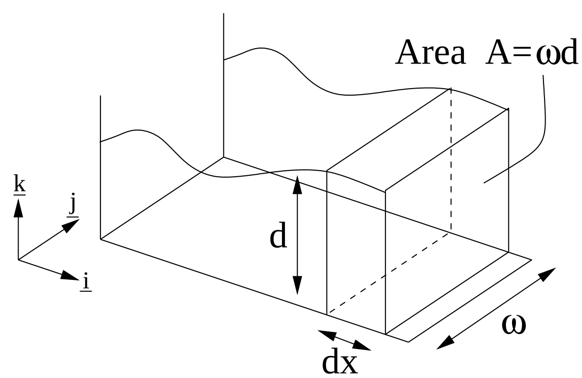
Question
Consider a channel with a rectangular cross-section. If the depth of the water is $d(x,t)$. where x measures distance along the channel and t is time.

(a) Show that d satisfies
$$\frac{\partial d}{\partial t} + \frac{\partial (vd)}{\partial x} = 0,$$
where $v(x,t)$ is the speed of the water in the channel.

(b) Find the analogous equation if the water flows in a circular pipe of radius R.

Answer

(a) Consider an element of the channel of length dx.



Conservation of mass for this element gives

$$\frac{dm}{dt} = -\int_{S+} \mathbf{j}\cdot\mathbf{n}\,ds + \int_{S-} \mathbf{j}\cdot\mathbf{n}\,ds$$

$m = \rho A dx \Rightarrow \mathbf{j} = \rho v(x)\mathbf{i}; \quad \mathbf{n} = \mathbf{i}$

Therefore $\rho dx \frac{\partial}{\partial t}(d\omega) = -\frac{\partial}{\partial x}(\rho v \omega d) dx$

So $\frac{\partial d}{\partial t} + \frac{\partial}{\partial x}(vd) = 0$

(b) Cross section:

PICTURE

$$\begin{aligned} \text{Area } A &= R^2\theta - (R-d)R\sin\theta \\ &= R^2\cos^{-1}\frac{R-d}{R} - R(R-d)\sqrt{1-\left(\frac{R-d}{R}\right)} \end{aligned}$$

The same procedure as above gives: $\dfrac{\partial}{\partial t}A(d) + \dfrac{\partial}{\partial x}(VA(d)) = 0$
(Substitution for A gives a messy equation)