QUESTION

Prove that, for any prime p, $\sqrt{p}$ is irrational.

ANSWER

Suppose $\sqrt{p}$ is rational, say $\sqrt{p} = \frac{a}{b}$ where $a, b \in Z$, and $\frac{a}{b}$ is cancelled to it's lowest terms, so that $\gcd(a, b) = 1$. We have $b\sqrt{p} = a$, so, on squaring, $b^2 p = a^2$. Thus $p|a^2$, so by question 2, $p^2|a^2$, say $a^2 = p^2 c$. Thus $b^2 p = p^2 c$, giving $b^2 = pc$. Hence $p|b^2$, and so by question 2, $p|b$. Thus $p|a$ and $p|b$, contrary to $\gcd(a, b) = 1$. Thus $\sqrt{p}$ is irrational, as claimed.