MA181 INTRODUCTION TO STATISTICAL MODELLING
CONTINUOUS DISTRIBUTIONS

Suppose X is a continuous variable with cumulative distribution function (cdf) $F(x) = P(X \leq x)$. We can differentiate $F(x)$ to obtain the probability density function (pdf) of X,

$$f(x) = \frac{dF(x)}{dx} = F'(x).$$

Inversely, we may write

$$F(x) = \int_{-\infty}^{x} f(u)du$$

or,more generally,

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(u)du.$$

If $h(X)$ is a function of X, we define its expected value to be

$$E[h(X)] = \int_x h(x)f(x)dx.$$

Example The random variable X has pdf

$$f(x) = \frac{3}{(x+1)^4}, 0 \leq x < \infty.$$

The cdf of X is therefore given by

$$F(x) = \int_0^3 \frac{1}{(u+1)^4}du = 1 - \frac{1}{(x+1)^3}.$$

Hence $P(X \leq 2) = 1 - \frac{1}{27} = \frac{26}{27},\ P(1 \leq X \leq 2) = \frac{26}{27} - \frac{7}{8}$ etc.

The mean of X is most easily found from

$$E(X+1) = 3\int_0^\infty \frac{1}{(x+1)^3}dx = -\frac{3}{2}\left[\frac{1}{(x+1)^2}\right]_0^\infty = \frac{3}{2},$$

so that $E(X) = \frac{3}{2} - 1 = \frac{1}{2}$. Similarly

$$E[(X+1)^2] = 3\int_0^\infty \frac{1}{(x+1)^2}dx = -3\left[\frac{1}{(x+1)^2}\right]_0^\infty = 3.$$

Since $E[(X+1)^2] = E(X^2) + 2E(X) + 1$ we have

$$\text{var}(X) = E[(X+1)^2] - 2E(X) - 1 - [E(X)]^2 = 3 - 2\left(\frac{1}{2}\right) - 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.$$

The exponential distribution If the number of events occurring during a fixed time interval follows a Poisson distribution, then the time interval T between two successive events follows an *exponential distribution*, which has the pdf

$$f(t) = \lambda e^{-\lambda t}, \ 0 < t < \infty,$$

where λ is a positive constant. The cdf of T is given by

$$F(t) = P(T \leq t = \lambda \int_0^t e^{-u}\, du = 1 - e^{-\lambda t}, \ 0 < t < \infty.$$

The mean of T is

$$E(T) = \lambda \int_0^\infty te^{-\lambda t}\, dt = \frac{1}{\lambda}$$

while $E(T^2) = \frac{2}{\lambda^2}$, so that $\text{var}(T) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

Moment generating function The probability generating function does not exist for a continuous distribution. A generating function that does exist, for both discrete and continuous distributions, is the moment generating function defined by

$$M(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} p(x), & x \text{ discrete,} \\ \int_x e^{tx} f(x)\, dx, & x \text{ continuous.} \end{cases}$$

Expanding e^{tx} leads, in either case, to

$$M(t) = \sum_{j=0}^{\infty} \frac{\mu_j' t^j}{j!}$$

so that μ_j' is the coefficient of $\frac{t^j}{j!}$ in the expansion of $M(t)$.

Example (Exponential) Suppose X follows an exponential distribution with pdf

$$f(x) = \lambda e^{-\lambda x},\ 0 < x < \infty.$$

Then

$$M(t) = \lambda \int_0^\infty e^{tx} e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx = \frac{\lambda}{\lambda - t} = \left[1 - \left(\frac{t}{\lambda}\right)\right]^{-1},\ t < \lambda.$$

This can be expanded to give $M(t) = 1 + \left(\frac{t}{\lambda}\right) + \left(\frac{t}{\lambda}\right)^2 + \ldots$ so that $\mu = \mu_1' = \frac{1}{\lambda}, \mu_2' = \frac{2}{\lambda^2},\ \mu_s' = \frac{6}{\lambda^3}, \ldots$
Consequently, $\text{var}(X) = \mu_2 = \mu_2' - \mu^2 = \frac{1}{\lambda^2}$.