MA181 INTRODUCTION TO STATISTICAL MODELLING
ELEMENTS OF PROBABILITY

1. An elementary result of an experiment is called an *outcome*.

2. The totality of the outcomes of an experiment constitutes the *sample space* S.

3. A set of outcomes is called an *event*. Thus an event is a subset of S. Events are often denoted by $A, B, C, \ldots$, or by $A_1, A_2, A_3, \ldots$.

Example - Birth months: Suppose an experiment consists of recording the birth months of my niece's first-born child. Then the sample space S consists of the twelve outcomes "born in January, "born in February",. . . ,"born in December". The event A, defined by the set {"born in March", "born in April", "born in May"}, may be defined as the event "born in Spring".

4. A function $P(.)$ defined on the sample space S is called a *probability function* if

(i) $P(S) = 1$,

(ii) $0 \leq P(A) \leq 1$ for any event A,

(iii) $P(A \text{ or } B) = P(A) + P(B)$ if A and B are mutually exclusive (i.e. they have no outcomes in common).

Note that the event "A or B" is the set of outcomes contained in A or B or both, and may also be written as $A \cup B$. Similarly $A \cap B$ or "A and B". is the set of outcomes common to both A and B. So, if A and B are mutually exclusive, then $A \cap B = \emptyset$.

If event A consists of the outcomes $w_1, w_2, \ldots, w_k$, then, by (iii),

$$P(A) = P(w_1) + P(w_2) + \ldots + P(w_k).$$

If, moreover, the outcomes of a finite sample space are all equally probable, then the probability of an event A is given by

$$P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Total number of outcomes}}.$$

An example is provided by a perfectly symmetrical die with faces numbered 1,2,3,4,5,6. All faces (outcomes) are equally probable with each having probability $\frac{1}{6}$ of being observed. Such a die is known as a fair die. If, further, A is the event "odd number". then $P(A) = \frac{3}{6} = \frac{1}{2}$.

Example - Birth months revisited: Suppose the months have the following probabilities of being the birth month:

$P(\text{Jan})= \frac{1}{12} - 0.04,$	$P(\text{Feb})= \frac{1}{12} - 0.02,$	$P(\text{Mar})= \frac{1}{12}$
$P(\text{Apr})= \frac{1}{12} + 0.01,$	$P(\text{May})= \frac{1}{12} + 0.02,$	$P(\text{Jun})= \frac{1}{12} + 0.02,$
$P(\text{Jul})=\frac{1}{12} + 0.02,$	$P(\text{Aug})= \frac{1}{12} + 0.03,$	$P(\text{Sep})= \frac{1}{12} + 0.01,$
$P(\text{Oct})= \frac{1}{12},$	$P(\text{Nov})= \frac{1}{12} - 0.02,$	$P(\text{Dec})= \frac{1}{12} - 0.03.$

Then $P(\text{born in Spring})= \frac{3}{12} + 0.01 + 0.02 = 0.28$.

Theorem 1 If A is an event in S, then $P(\overline{A}) = 1 - P(A)$.

Proof Now $A \cup \overline{A} = S$ while $A \cap \overline{A} = \phi$. Hence, by (iii) above,

$$1 = P(S) = P(A \cup \overline{A}) = P(A) + P(\overline{A}),$$

so that $P(\overline{A}) = 1 - P(A)$.

Theorem 2 If A and B are two events in S, then $P(A) = P(A \cap B) + P(A \cap \overline{B})$.

Proof In the diagram, $A \cap B$ is hatched horizontally, while $A \cap \overline{B}$ is hatched vertically. Now

$$A = (A \cap B) \cup (A \cap \overline{B}) \text{ and } (A \cap B) \cap (A \cap \overline{B}) = \emptyset.$$

Hence, by (iii) above,

$$P(A) = P(A \cap B) + P(A \cap \overline{B}).$$

5. Theorem 3 (Addition rule) For any two events A and B in S,

$$P(A \text{ or } B) = P(A) + P(B) = P(A \text{ and } B).$$

Proof In the diagram, event A is hatched diagonally while $\overline{A} \cap B$ is hatched horizontally. Now

$$A \cup B = A \cup (\overline{A} \cap B) \text{ and } A \cap (\overline{A} \cap B) = \emptyset.$$

Hence

$$\begin{aligned} P(A \cup B) &= P(A) + P(\overline{A} \cap B), \text{ by (iii)}, \\ &= P(A)_+ P(B) = P(A \cap B), \text{ by Theorem 2.} \end{aligned}$$

Example Let A be the event "even number" and B the event "number> 3" when a fair die is thrown. Then $P(A) = P(B) = \frac{3}{6}$ and $P(A \text{ and } B) = \frac{2}{6}$. Hence $P(A \text{ or } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6}$, i.e. $P(\{2, 4, 5, 6\}) = \frac{2}{3}$.

6. Multiplication rule Let A and B be two events in S. Then the *conditional probability* of A given B is defined as $P(A|B) = P(A \text{ and } B)/P(B)$ so long as $P(B) > 0$.

Similarly $P(B|A) + P(A \text{ and } B)/P(A)$, so long as $P(A) > 0/$

Note that $P(A|B)$ represents the probability of A in the reduced sample space defined by B.

When these definitions are written in the form

$$P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B),$$

this is know as the *multiplication rule*.

Example: A bag contains three black balls and three white balls. I draw two balls from the bag without replacement. The probability that I draw two white balls is

$$P(\text{first ball is white})P(\text{second ball is white—first is white}) = \frac{3}{6} \times \frac{2}{5} = \frac{1}{5}.$$

Example: I shake a fair die. Then

$$\begin{aligned} P(\text{number>3—even number}) &= \frac{P(\text{number>3 and even number})}{P(\text{even number})} \\ &= \frac{P(\{4, 6\})}{P(\{2, 4, 6\})} = \frac{2}{3}. \end{aligned}$$

Example - Birth months revisited: Now

$$
\begin{aligned}
P(\text{Born in may—born in Spring}) &= \frac{P(\text{born in May and born in Spring})}{P(\text{born in Spring})} \\
&= \frac{\frac{1}{12} + 0.02}{\frac{1}{4} + 0.03} \\
&= \frac{31}{84} = 0.369
\end{aligned}
$$

Example: Suppose a person is chosen at random from the population, that F is the event that he or she speaks French and G is the event that he or she speaks German. Suppose, moreover, that $P(F) = \frac{1}{100}$, $P(G) = \frac{1}{500}$ and $P(F|G) = \frac{2}{3}$. Then

$$
P(G|F) = \frac{P(F \text{ and } G)}{P(F)} = \frac{P(F|G)P(G)}{P(F)} = \frac{\frac{2}{3} \cdot \frac{1}{500}}{\frac{1}{100}} = \frac{2}{15}.
$$

Theorem 4 For any two events A and B in SW,

$$
P(A) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B}).
$$

[This is a simple case of the theorem of total probability.]

Proof From Theorem 2,

$$
P(A) = P(A \cap B) + P(A \cap \overline{B}).
$$

However,

$$
P(A \cap B) = P(B)P(A|B) \text{ and } P(A \cap \overline{B} = P(\overline{B})P(A|\overline{B}).
$$

Consequently, after substitution, we have the result.

7. Events A and B are said to be *independent* if

$$
P(A \text{ and } B) = P(A)P(B).
$$

Thus $P(A|B) = P(A)$, so long as $P(B) > 0$, and $P(B|A) = P(B)$, as long as $P(A) > 0$.

Example: On a fair die, let A be the event "number$\leq$2" and B the event "even number". Then

$$P(A) = \frac{1}{3},\ P(B) = \frac{1}{2},\ P(A \text{ and } B) = \frac{1}{6}.$$

Hence A and B are independent.

Example: Two dice are shaken together. If A is an event on one of them and B an event on the other, then A and B are assumed to be independent. Consequently the probability of two sixes is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example: I shake two fair dice. What is the probability that the total shown is six?

P(total is 6)=P(first die shows 1 and second 5)+P(first die shows 2 and second 4)+…+P(first die shows 5 and second 1)

=P(first die shows1).P(second shows 5)+…+P(first die shows 5).P(second shows 1)

$= \left(\frac{1}{6} \cdots \frac{1}{6}\right) + \left(\frac{1}{6} \cdot \frac{1}{6}\right) + \ldots + \left(\frac{1}{6} \cdot \frac{1}{6}\right) = \frac{5}{36}$.

Example: A machine produces metal rods for cars, of which $\frac{1}{100}$ are too long while $\frac{1}{60}$ are dent, these two faults occurring independently. What is the probability of a faulty rod? A rod is faulty if it is too long or bent (or both). Hence

$$P(\text{faulty rod}) = \frac{1}{100} + \frac{1}{60} - \left(\frac{1}{100} \times \frac{1}{60}\right) = \frac{53}{2000} = 0.0265.$$

8. The above results can be extended to more than two events. So, for three events A, B and C in S,

(i) $P(A \text{ or } B \text{ or} C) = P(A)+P(B)+BP(C)-P(A \text{ and } B)-P(A \text{ and } C)-P(B \text{ and } C) + P(A \text{ and } B \text{ and } C)$,

(ii) $P(A \text{ and } B \text{ and } C) = P(A)P(B|A)P(C|A \text{ and } B)$ etc.

(iii) $A < B$ and C are independent if

$$P(A \text{ and } B) = P(A)P(B),\ P(A \text{ and } C) = P(A)P(C),$$
$$P(B \text{ and } C) = P(B)P(C) \text{ and } P(A \text{ and } B \text{ and } C) = P(A)P(B)P(C).$$

9. Bayes' formula: Let A be an event in S, and let $B_1, B_2, \ldots, B_k$ be a set of mutually exclusive and exhaustive events in S. Then

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}.$$

This is known as *Bayes' formula* after the Reverend Thomas Bayes (1702-1761).

Derivation From the multiplication rule, we have

$$P(B_i|A) = P(A|B_i)P(B_i)/P(A).$$

Since the B_I's are mutually exclusive and exhaustive,

$$\sum_{j=1}^{k} P(B_j|A) = 1$$

so that

$$P(A) = \sum_{j=1}^{k} P(A|B_j)P(B_j)$$

and the result follows.

Example: One bag contains two white balls, a second contains one white and one black and a third contains two black balls. A bag is chosen at random and a ball drawn from it, which is found to be white. What is the probability that the remaining ball in the bag is also white?

Let the choices of the bags be denoted by B_1, B_2 and B_3, and let A be the event "a white ball is drawn". Then

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

and

$$P(A|B_1) = 1, \; P(A|B_2) = \frac{1}{2}, \; P(A|B_3) = 0.$$

Consequently, from Bayes' formula we have

$$P(B_1|A) = \frac{1 \times \frac{1}{3}}{\left(1 \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right)}, \; P(B_2|A) = \frac{1}{3} \text{ and } P(B_3|A) = 0$$

Example Mum, John and Sue share the weekly washing-up duties. Mum washes up 14 times a week, Sue 5 times and John twice, the specific occasions being decided by drawing lots. When washing up, Mum breaks a crock with probability $\frac{1}{200}$, Sue with probability $\frac{1}{150}$ and John with probability $\frac{1}{40}$. Dad is reading the newspaper when he hears a crock break. What is the probability that John is the culprit?

Let A, B_1, B_2, B_3 be the events

A: "crock breaks",
B_1: "Mum washing up",
B_2: "John washing up",
B_3: "Sue washing up".

Now $P(B_1) = \frac{14}{21}$, $P(B_2) = \frac{2}{21}$, $P(B_3) = \frac{5}{21}$ while $P(A|B_1) = \frac{1}{200}$, $P(A|B_2) = \frac{1}{40}$ and $P(A|B_3) = \frac{1}{150}$. From Bayes' formula, we therefore find

$$P(B_2|A) = \frac{\frac{1}{40} \times \frac{2}{21}}{\left(\frac{1}{200} \times \frac{14}{21}\right) + \left(\frac{1}{40} \times \frac{2}{21}\right)\left(\frac{1}{150} \times \frac{5}{21}\right)} = \frac{15}{46}.$$