Question

Briefly describe the main features of a EUROPEAN AVERAGE STRIKE option, explaining in particular how average strike options differ from European average rate and European vanilla options.

YOU MAY ASSUME that it the payoff of a path-dependent option is dependent on the quantities $S(T)$ and

$$\int_0^T f(S(\tau), \tau)\, d\tau$$

then the independent variable

$$I = \int_0^t f(S(\tau), \tau)\, d\tau$$

satisfies the stochastic differential equation

$$dI = f(S, t)dt.$$

Use this fact and Ito's lemma (which you may use without proof) to show that the value $V = V(S, I, t)$ of such an option satisfies the partial differential equation

$$V_t + f(S, t)V_I + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

where, as usual, V, S, t, σ and r denote respectively the option value, the asset price, time, the volatility and the interest rate.

Show that in the case of the continuously sampled arithmetic average strike the governing differential equation may be reduced to one involving only two independent variables by setting

$$V(S, R, t) = IW(R, t)$$

where

$$R = \frac{S}{\int_0^t S(\tau)\, d\tau}$$

and

$$R = \frac{S}{I}.$$

Answer
For a European Vanilla with strike E, the payoff is $\max(S-E,0)$ (CALL) or $\max(E-S,0)$ (PUT), which depends only on S AT EXPIRY. The payoff for a Euro Av Strike is the same as this, except that E is replaced by the average of S over the contract period. So for an average strike thee os no "Exercise" price. For an AVERAGE RATE option the same is true, but now S is replaced by the average.
Now assume

$$\begin{aligned} V &= V(I,S,t) \\ I &= \int_0^t f(S(\tau),\tau)\,d\tau \\ dI &= f(S,t)dt \end{aligned}$$

Consider the portfolio $\Pi = V - \Delta S$.
Then as usual $d\Pi = dV - \Delta dS$.

$$\begin{aligned} \text{Also dS} &= \sigma S dt + \sigma D dX \quad \text{(as usual)} \\ \text{and dI} &= f(S,t)dt \end{aligned}$$

Also in the usual way $dX \sim \sqrt{dt}$ as $dt \to 0$
$\Rightarrow$ in the limit $dS^2 = \sigma^2 S^2 dt$
Taylor $\Rightarrow$

$$dV = V_S dS + V_t dt + V_I dI + \frac{1}{2}V_{SS}dS^2$$

$$\begin{aligned} d\Pi &= V_S dS + V_t dt + V_I dI + \frac{1}{2}V_{SS}\sigma^2 S^2 dt - \Delta(\sigma S dt + \sigma S dX) \\ &= V_S(\sigma S dt + \sigma S dX) + V_t dt + V_I f(S,t)dt \\ &\quad + \frac{1}{2}V_{SS}\sigma^2 S^2 st - \Delta\sigma S dt - \Delta\sigma S dX \end{aligned}$$

As usual, the randomness vanishes if we choose $\Delta = V_S$

$$\begin{aligned} \Rightarrow d\Pi &= V_S\sigma S dt + V_t dt + V_I f dt + \frac{1}{2}V_{SS}\sigma^2 S^2 dt - V_S\sigma S dt \\ &= \left(V_t + f(S,t)V_I + \frac{1}{2}V_{SS}\sigma^2 S^2\right) dt \end{aligned}$$

Now by the usual arbitrage argument

$$d\Pi = r\Pi dt = r(V - \Delta S)dt = r(V - SV_S)dt$$

$$\Rightarrow V_t + f(S,t)V_I + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - 0.$$

Now the average is $A(t) = \frac{1}{t} \int +0^t S(\tau)\, d\tau$
and we are to try $V = IW, \quad (R + S/I)$.

$$\begin{aligned} \Rightarrow V_t &= IW_t \\ V_I &= W - \frac{S}{I} W_R \\ V_S &= \frac{I}{I} W_R \\ V_{SS} &= \frac{1}{I} W_{RR} \end{aligned}$$

$$\Rightarrow IW_t + f(S,t)(W - SW_R/I) + \frac{1}{2}\frac{\sigma^2 S^2}{I} W_{RR} + rSW_R - rIW = 0$$

but $R = S/I$

$$\Rightarrow W - t + \frac{f}{I}(W - RW_R)\frac{1}{2}\sigma^2 R^2 W_{RR} + rRW_R - rW = 0$$

Now $f(S,t) = IR$ since $I = \int_0^t S(\tau)\, d\tau$.

$$\begin{aligned} \Rightarrow W_t + \frac{1}{2}\sigma^2 R^2 W_{RR} - R^2 W_R + RW + rRW_R - rW &= 0 \\ \Rightarrow W_t + \frac{1}{2}\sigma^2 R^2 W_{RR} + R(r - R)W_R - (r - R)W &= 0 \end{aligned}$$