Question

(a) An up-and-out barrier Put option is identical to a European Vanilla Put, save for the fact that if any time during the life of the option the asset price exceeds the barrier B, the option instantly becomes (and remains) worthless. Assuming that the underlying asset pays no dividends, explain briefly why the fair price V of the option must satisfy the boundary value problem

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0, \quad (S < B),$$

$$V(B,t) = 0, \quad V(S,T) = \max(E - S, 0), \quad (S < B).$$

(Here as usual the asset value, strike price, volatility and interest rate are denoted by S, E, σ and r respectively.)

(b) Assume now that for a particular up-and-out Put B¿E. Show that if U(S,t) satisfies the Black-Scholes equation and V is defined by

$$U(S,t) = S^n V(\eta, t), \quad \left(\eta = \frac{K}{S}\right)$$

where K is an arbitrary constant, then V also satisfies the Black-Scholes equation provided n takes a specific value (which you should determine).

Hence or otherwise show that the fair value of and up-and-out barrier Put is given by

$$V = P_{BS}(S,t) - \left(\frac{S}{B}\right)^{1-2r/\sigma^2} P_{BS}(B^2/S, t)$$

where P_{BS} denotes the value of a European Vanilla Put.

Answer

(a) An up-and-out barrier option is just like a normal Put until the barrier is reached, and so satisfies Black-Scholes.

$$\Rightarrow V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \quad (S < B)$$

As soon as $S = B$ it is worthless

$$\Rightarrow V(B,t) = 0$$

(as this must apply instantaneously and for all time.)

Provided $S < B$ the payoff is just that for a Euro Vanilla Put and thus

$$V(S,T) = \max(\mathrm{E} - \mathrm{S}, 0).$$

(b) Now $U(S,t) = S^n V(\eta, t)$, $(\eta = K/S)$

$$\begin{aligned}
\Rightarrow U_t &= S^n V_t \\
U_S &= nS^{n-1}V - KS^{n-2}V_\eta = nS^{n-1}V - \eta S^{n-1}V_\eta \\
U_{SS} &= n(n-1)S^{n-2}V - nS^{n-2}\eta V_\eta - (n-2)S^{n-2}\eta V_\eta \\
&\quad +\eta^2 S^{n-2} V_{\eta\eta}
\end{aligned}$$

Now U satisfies Black-Scholes so

$$U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + rSU_S - rU = 0$$

$$\begin{aligned}
\Rightarrow \quad & S^n V_t + \frac{1}{2}\sigma^2 S^2 \Big[n(n-1)S^{n-2}V - nS^{n-2}\eta V_\eta \\
& -(n-2)\eta S^{n-2} V_\eta + \eta^2 S^{n-2} V_{\eta\eta} \Big] \\
& +rS \Big[nS^{n-1}V - \eta S^{n-1} V_\eta \Big] - rS^n V \\
& = 0
\end{aligned}$$

Thence

$$V_t + \frac{1}{2}\sigma^2 \Big[(n^2 - n)V - n\eta V_\eta - \eta(n-2)V_\eta + \eta^2 V_{\eta\eta} \Big]$$

$$+rn[V] - r\eta V_\eta - rV = 0$$

so that

$$V_t + \frac{1}{2}\sigma^2\eta^2 V_{\eta\eta} + V_{eta}\left[-\frac{1}{2}\sigma^2 n\eta - r\eta - \eta(n-2)\frac{1}{2}\sigma^2\right]$$

$$+V\left[\frac{\sigma^2}{2}(N^2 - n) + r(n-1)\right] = 0$$

$\Rightarrow$

$$V_t + \frac{1}{2}\sigma^2\eta^2 V_{\eta\eta} + \eta V_\eta\left[-r - \sigma^2[n-1]\right]$$

$$+(n-1)\left[r + \frac{1}{2}n\sigma^2\right]V = 0$$

To get Black-Scholes out of this we need

$$-r - \sigma^2(n-1) = r$$

$$\Rightarrow n = 1 - \frac{2r}{\sigma^2}$$

But then

$$(n-1)\left[r + \frac{1}{2}n\sigma^2\right] = -\frac{2r}{\sigma^2}\left[r + \frac{1}{2}\sigma^2\left(1 - \frac{2r}{\sigma^2}\right)\right] = -r$$

Thus with $n = 1 - 2r/\sigma^2$, V satisfies the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2\eta^2 V_{\eta\eta} + r\eta V_\eta - rV = 0.$$

Since Black-Scholes is linear we may add solution. So consider a solution of the form

$$V = P_{BS}(S,t) + AS^n P_{BS}(K/S,t)$$

where A is a constant and n is chosen as above.

We have:-

(i) This satisfies Black-Scholes $\forall A$, $\forall K$

(ii) We need $V(B,t) = 0$. Thus

$$0 = P_{BS}(B,t) + AB^n P_{BS}(K/B,t).$$

Clearly this condition holds if we set $K = B^2$ and $A = -B^{-n}$

Thus

$$V = P_{BS}(S,t) - \left(\frac{S}{B}\right)^n P_{BS}(B^2/S,t), \quad (n = 1 - 2r/\sigma^2)$$

Finally we must check the payoff.

At expiry

$$\begin{aligned} V(S,T) &= P_{BS}(S,T) - \left(\frac{S}{B}\right)^n P_{BS}(B^2/S,T) \\ &= \max(\mathrm{E-S},0) - 0 \end{aligned}$$

since if $E < B$ the second term is $\max(E - B^2/S, 0) = 0$.