Question

(a) Draw payoff diagrams (from the holder's point of view) for a European Vanilla Call and a European Vanilla Put, both with strike price E and expiry T. On financial grounds, would you expect American Vanilla Calls and Puts to be more or less expensive than their European equivalents?

(b) A POWER OPTION is an option whose payoff is given by

$$V(S,T) = AS^n$$

where A and n are constants. Show that the Black-Scholes equation admits solutions of the form

$$V(S,t) = g(t)S^n$$

provided that $g(t)$ satisfies the ordinary differential equation

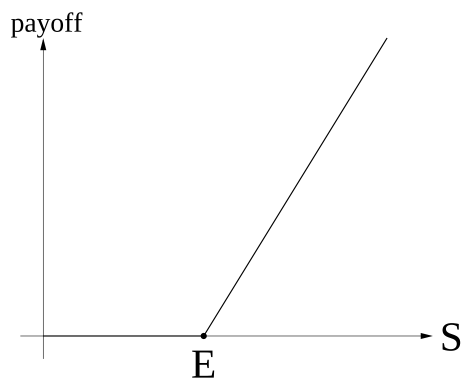
$$\frac{dg}{dt} + \left(\frac{1}{2}\sigma^2 n + r\right)(n-1)g = 0.$$

Hence or otherwise find the fair value for a power option. What financial product does the option become in the special cases $n = 0$ and $n = 1$?

Answer

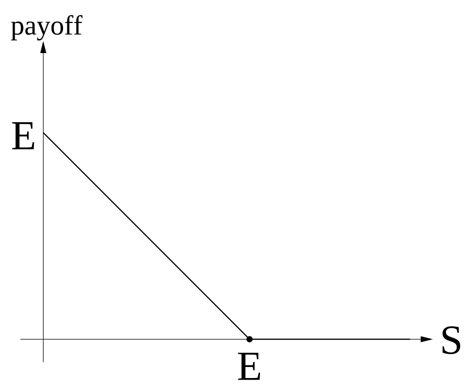
(a) Payoff for Euro Vanilla Call is $\max(S-E,0)$

$\Rightarrow$ Payoff



Payoff for Euro Vanilla Put is $\max(E-S,0)$

$\Rightarrow$ Payoff



The ability to exercise early that is passed by an American option is evidently a great advantage to have. $\Rightarrow$ American Vanilla should be more expensive than their European counterparts.

(b) We have $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$

Now try

$$V(S,t) = g(t)S^n.$$

$$\begin{aligned} \Rightarrow g'S^n + \frac{1}{2}\sigma^2 S^2 n(n-1)S^{n-2}g + rSgnS^{n-1} - rgS^n &= 0 \\ g' + \frac{1}{2}\sigma^2 n(n-1)g + rgn - rg &= 0 \end{aligned}$$

$$\begin{aligned} g' + \left[\frac{1}{2}\sigma^2 n(n-1) + rn - r\right] g &= 0 \\ \text{i.e.} \quad \mathrm{g' + n(n-1)\left[\frac{1}{2}n\sigma^2 + r\right] g} &= 0. \end{aligned}$$

Thus

$$g(t) = \tilde{A}e^{(1-n)\left[r+\frac{1}{2}n\sigma^2\right]t}$$

where $\tilde{A}$ is an arbitrary constant, and so in all

$$V = S^n\tilde{A}e^{(1-n)\left[r+\frac{1}{2}n\sigma^2\right]t}.$$

Now we know that at expiry $V(S,T) = AS^n$ where A and n are given. Thus

$$\begin{aligned} AS^n &= S^n\tilde{A}e^{(1-n)\left[r+\frac{1}{2}n\sigma^2\right]T} \\ \Rightarrow \tilde{A} &= Ae^{-(1-n)\left[R+\frac{1}{2}n\sigma^2\right]T}. \end{aligned}$$

and so the value of the option is

$$V = Ae^{(1-n)\left[r+\frac{1}{2}n\sigma^2\right](t-T)}S^n.$$

When $n=0$, $V = Ae^{r(t-T)}$ and the option is just the same as an initial amount A of cash growing at the risk-free rate.

When $n=1$, $V = AS$ and the option is just the same as holding A of the underlying asset.