Question

(a) On the last day of 1999, two investors (Asif and Barbara) decided to spend money in buying options. On this day, the share price of BOOTS was 495p, the share price of BP AMOCO was 495p and the share price of SHELL was 438p. The expiry date for all of the options that they bought was April 13th. Options were available as follows

Option	Strike Price	Available for
BOOTS PUTS	460p	20p
BOOTS CALLS	460p	60p
BP PUTS	500p	37p
BP CALLS	500p	34p
SHELL PUTS	460p	45p
SHELL CALLS	460p	20p

Asif bought 1000 BOOTS calls, 1000 BP calls and 500 SHELL puts, and Barbara bought 1000 BOOTS puts, 2000 BP puts and 2000 SHELL calls.

Assuming that there was no bid/ask spread and no dealing charges, and that on April 13th 2000 the share prices were given by:

BOOTS	555p
BP AMOCO	410p
SHELL	500p

determine how much Asif and Barbara paid for their options, and what the total profit or loss was for each investor after expiry.

(b) Now YOU MAY ASSUME that small charges df in the function $f(S,t)$ are related to small changes in S and t by Taylor's theorem and that the asset prices S of a share follows the lognormal random walk

$$dS = rSdt + \sigma SdX$$

where X is a random variable, r and σ are constants, and $dX^2 \to dt$ as $dt \to 0$.

By considering a portfolio $\Pi = V - \Delta S$ (where Δ is to be determined), show that the fair value V of an option satisfies the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

Answer

(a) As instructed we assume no dealing costs or bid/ask spreads. Proceeding with the given amounts and prices, we find that they SPENT

(A)

$$(1000)(0.60) + (1000)(0.34) + (500)(0.45) = £1165$$

(B)

$$(1000)(0.20) + (2000)(0.37) + (2000)(0.20) = £1340$$

Now at expiry we have Boots$= 555p$, BP$= 410p$, Shell$= 500p$.

The payoffs are thus:-

Boots:-	Puts worthless	
	Calls$= 555 - 460$	$= £0.95$
BP:-	Puts$= 500 - 410$	$= £0.90$
	Calls worthless	
Shell:-	Puts worthless	
	Calls$= 500 - 460$	$= £0.40$

At expiry therefore the positions are

(A)

$$1000 \times 0.95 + 1000 \times 0 + 500 \times 0 = £950$$

(B)

$$1000 \times 0 + 2000 \times 0.90 + 2000 \times 0.40 = £2600$$

Thus ASIF lost £215, BARBARA gained £1260.

(b)

We have $dS = rSdt + \sigma dX$ and

$$df = f_S dS + f_t dt + \frac{1}{2} F_{SS} dS^2 + f_{St} dt dS + \frac{1}{2} f_{tt} dt^2 + \cdots$$

Now consider the given portfolio $\Pi = V - \Delta S$.

Then

$$d\Pi = dV - \Delta dS.$$

Now to find dV we have

$$\begin{aligned} dV &= V_S dS + V_t dt + \frac{1}{2} V_{SS} dS^2 + V_{St} dt dS + \frac{1}{2} V_{tt} dt^2 + \cdots \\ &= V_S (rSdt + \sigma S dX) + V_t dt \\ &\quad + \frac{1}{2} V_{SS} (r^2 S^2 dt^2 + \sigma^2 S^2 dX^2 + 2r\sigma S^2 dt dX) \\ &\quad + V_{St} dt (rSdt + \sigma S dX) + \frac{1}{2} V_{tt} dt^2 + \cdots \end{aligned}$$

But since we have $dX^2 \to dt$ as $dt \to 0$

$$\Rightarrow dV = V_S rSdt + V_S \sigma S dX + V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dX^2$$

$$+ O(dXdt) + O(dt^2) + \cdots$$

So $dV = rSV_s dt + \sigma S V_S dX + V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dt$ to leading order.

Thus

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= rSV_S dt + \sigma S V_S dX + V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dt - \Delta dS \\ &= rSV_S dt + \sigma S V_S dX + V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dt \\ &\quad - \Delta rSdt - \Delta \sigma S dX \end{aligned}$$

The randomness may now be eliminated by setting $\Delta = V_s$, giving

$$d\Pi = V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dt.$$

The usual arbitrage argument now says that this must be equivalent to putting money in the bank,

So

$$d\Pi = r\Pi dt = rdt(V - \Delta S) = rdt(V - V_S S)$$

$$\Rightarrow V_t dt + \frac{1}{2}\sigma^2 S^2 V_{SS} dt = rVdt - rSV_S dt$$

$$\Rightarrow V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$