Question

For each of the following quadratic forms Q on $\mathbf{R}^n$ find the nature of the critical point at the origin. Describe also the geometry of the level set $Q^{-1}(1)$:

$n=2$; $Q(x,y)=$

(i) x^2+y^2-xy **(ii)** $9x^2-12xy+4y^2$

$n=3$; $Q(x,y,z)=$

(iii) $x^2+y^2+z^2+xy+yz+zx$ **(iv)** $x^2+y^2+z^2-xy$

(v) $2x^2+y^2-4xy-4yz$ **(vi)** $3x^2+y^2+z^2-2xy+2xz-2yz$

In each case (i) – (vi) say whether the addition of higher order terms to Q could change the nature of the level sets close to the origin.

Answer

(i) x^TAx **with** $A=\begin{pmatrix} 1 & -frac12 \\ -\frac{1}{2} & 1\end{pmatrix}$.

Eigenvalues

$$\begin{aligned}(1-\lambda)^2-\frac{1}{4} &= 0\\ \Rightarrow \lambda^2-2\lambda+\frac{3}{4} &= 0\\ \Rightarrow \lambda_1\lambda_2 &= \mathbf{det}A=\frac{3}{4}>0\\ \lambda_1+\lambda_2 &= 2>0\end{aligned}$$

So as λ_1, λ_2 both >0: minimum ($Q^{-1}(1)$: ellipse).

[* $Q=(x_1-\frac{1}{2}x_2)^2+\frac{3}{4}x_2^2$, hence minimum.]

(ii)

$A=\begin{pmatrix} 9 & -6 \\ -6 & 4\end{pmatrix}$.

Eigenvalues 0, 13: "trough" ($Q^{-1}(1)$: pair of lines).

[* $Q=(3x-2y)^2$.]

(iii)

$A=\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1\end{pmatrix}$.

Eigenvalues

$$\begin{aligned}&(1-\lambda)((1-\lambda)^2-\frac{1}{4})-\frac{1}{2}(\frac{1}{2}(1-\lambda)-\frac{1}{4})+\frac{1}{2}(\frac{1}{4}-\frac{1}{2}(1-\lambda))\\ =\ &(1-\lambda)(\lambda^2-2\lambda+\frac{3}{4})-\frac{1}{2}(\frac{1}{4}-\frac{1}{2}\lambda)+\frac{1}{2}(-\frac{1}{4}+\frac{1}{2}\lambda)\end{aligned}$$

$$
\begin{aligned}
&= -\lambda^3 + 3\lambda^2 - \frac{9}{4}\lambda + \frac{1}{2} \\
&= -\frac{1}{4}(4\lambda^3 - 12\lambda^2 + 9\lambda - 2) \\
&= 0
\end{aligned}
$$

Clearly $\lambda = \frac{1}{2}$ is an eigenvalue, so

$$
\begin{aligned}
(2\lambda - 1)(2\lambda^2 - 5\lambda + 2) &= 0 \\
(2\lambda - 1)^2(\lambda - 2) &= 0
\end{aligned}
$$

Thus eigenvalues are $\frac{1}{2}$, $\frac{1}{2}$, 2: all positive.

$\Rightarrow$ minimum ($Q^{-1}(1)$: ellipsoid).

(iv)

$$
A = \begin{pmatrix} 17 - \frac{1}{2} & 0 & \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Eigenvalues $\lambda = 1$ and two other positives (see (i) above) $\Rightarrow$ minimum

[* $Q = (x_1 - \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 + x_3^2$] $\leftarrow$ ($Q^{-1}(1)$: ellipsoid.]

(v)

$$
A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}.
$$

Eigenvalues

$$
\begin{aligned}
(2 - \lambda)((1 - \lambda)(-\lambda) - 4) + 2(2\lambda) &= 0 \\
(2 - \lambda)(\lambda^2 - \lambda - 4) + 4\lambda &= 0 \\
-\lambda^3 + 3\lambda^2 + 6\lambda - 8 &= 0 \ : \ \lambda = 1 \\
-(\lambda - 1)(\lambda^2 - 2\lambda - 8) &= 0
\end{aligned}
$$

The quadratic equation $\lambda^2 - 2\lambda - 8 = 0$ has roots $\lambda = -2, 4$, $\Rightarrow$ saddle ($Q^{-1}(1)$: hyperboloid (1 sheet)).

(vi)

$$
A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}
$$

$\lambda = 0, 1, 4$. **Degenerate critical point (line of zeros). ($Q^{-1}(1)$: elliptical tube).**

(* indicates short cuts)

[Addition of higher order terms can affect conclusion only when $\det A = 0$ ($\exists$ zero eigenvalue), namely cases (ii), (vi).]