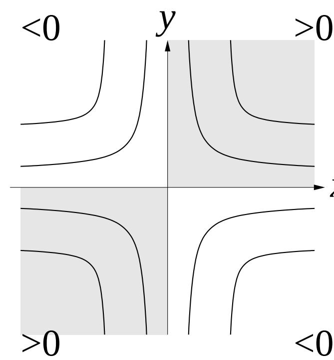
Question

For each of the following functions $f : \mathbf{R}^2 \to \mathbf{R}$ find the critical points and decide if they are nondegenerate or degenerate. Classify the nondegenerate critical points as maxima, minima or saddles. Sketch contours for f, taking care to include those which pass through the critical points.

$f(x,y) =$	**(i)** $xy(x^4+y^4+1)$		**(iv)**	$y^2-3yx^2+2x^4$
	(ii) x^3+y^2-3x		**(v)**	x^3-3xy^2
	(iii) $\sin x+\sin y+\cos(x+y)$		**(vi)**	$x^4+y^4-2(x^2+y^2)$

Answer

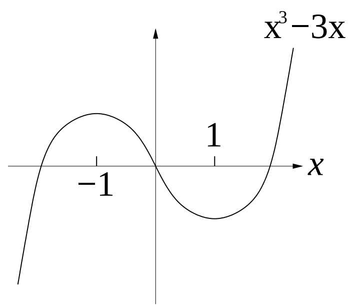
(i) $xy(x^4+y_1^4)$**. Vanishes on axes** $x=0$**,** $y=0$ **only.**

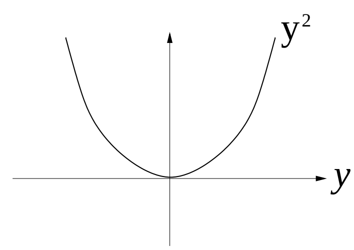


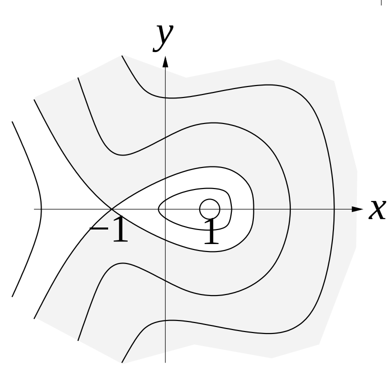
Saddle point at $(0,0)$ **(Hessian matrix** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$**).**

$f(x,y) = xy \times$ (**positive quantity which increases as** $\|(x,y)\| \to \infty$).

(ii) $(x^3-3x)+y^2$**:**







Hence minimum at $(1,0)$**, saddle at** $(-1,0)$**.**

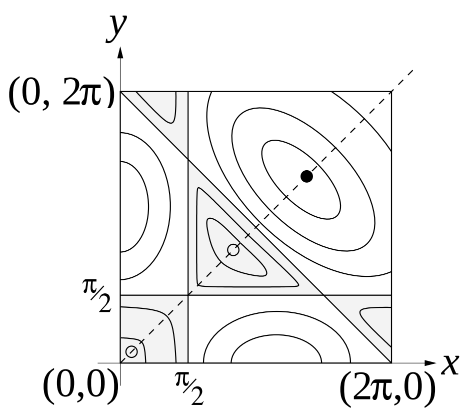
Dots where $f>2$**, white where** $f<2$**.**

(iii) $\left.\begin{aligned}\frac{\partial f}{\partial x} &= \cos x - \sin(x+y)\\ \frac{\partial f}{\partial y} &= \cos y - \sin(x+y)\end{aligned}\right\}$ **vanish when** $\begin{aligned}\cos x &= \cos y\\ x &= 2n\pi \pm y\end{aligned}$

Then

$x=y$ **gives** $x=y=\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$**; and** $\frac{\pi}{2}$**mod**2π**.**

$x=-y$ **gives** $x=-y=\frac{\pi}{2}$, $\frac{3\pi}{2}$**mod**2π**.**



o=max, •=min, ×=saddle

($f(x,y)$ **symmetric about the line** $x=y$**.)**

Note also

$$\begin{aligned} x = \frac{\pi}{2} &\Rightarrow f=1\\ y = \frac{\pi}{2} &\Rightarrow f=1\\ \text{and } x+y=2\pi &\Rightarrow f=1 \end{aligned}$$

Hessian matrices as follows

$$\left(\frac{\pi}{6},\frac{\pi}{6}\right): \begin{pmatrix} -1 & -\frac{1}{2}\\ -\frac{1}{2} & 1\end{pmatrix} \text{ max} \qquad \left(\frac{5\pi}{6},\frac{5\pi}{6}\right): \begin{pmatrix} -1 & -\frac{1}{2}\\ -\frac{1}{2} & -1\end{pmatrix} \text{ max}$$

$$\left(\frac{3\pi}{2},\frac{3\pi}{2}\right): \begin{pmatrix} 2 & 1\\ 1 & 2\end{pmatrix} \text{ min} \qquad \left(\frac{\pi}{2},\frac{\pi}{2}\right): \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \text{ saddle}$$

$$\left(\frac{\pi}{2},\frac{3\pi}{2}\right): \begin{pmatrix} -2 & -1\\ -1 & 0\end{pmatrix} \text{ saddle} \qquad \left(\frac{3\pi}{2},\frac{\pi}{2}\right): \begin{pmatrix} 0 & -1\\ -1 & -2\end{pmatrix} \text{ max}$$

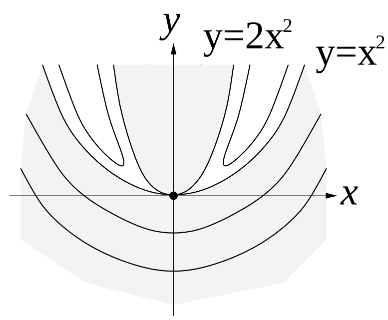
Dotted region is where $f>1$**, (max=**$\frac{3}{2}$**)**

White region is where $f<1$**, (min=**$-\frac{3}{2}$**)**

Note that $f(x,0)=\sqrt{2}\sin(x+\frac{\pi}{4})$**.**

(iv)

$$\begin{aligned} y^2 - 3yx^2 + 2x^4 &= \left(y - \frac{3}{2}x^2\right)^2 \\ &= (y-x)(y-2x^2) \\ &= 0 \\ \textbf{when } \frac{1}{2}x^2 &= \pm\left(y = \frac{3}{2}x^2\right) \\ \Rightarrow y &= x^2 \textbf{ or } 2x^2 \end{aligned}$$

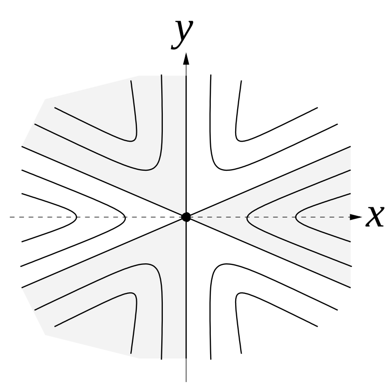


The only critical point is $(0,0)$**.**

Dotted region: $f > 0$

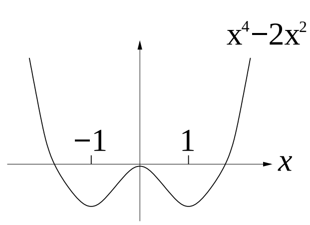
White region: $f < 0$**.**

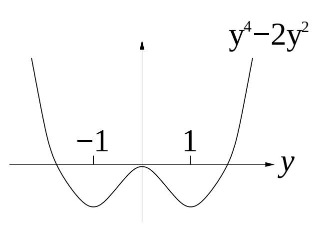
(v) $x^3 - 3xy^2 = x(x - \sqrt{3}y)(x + \sqrt{3}y)$



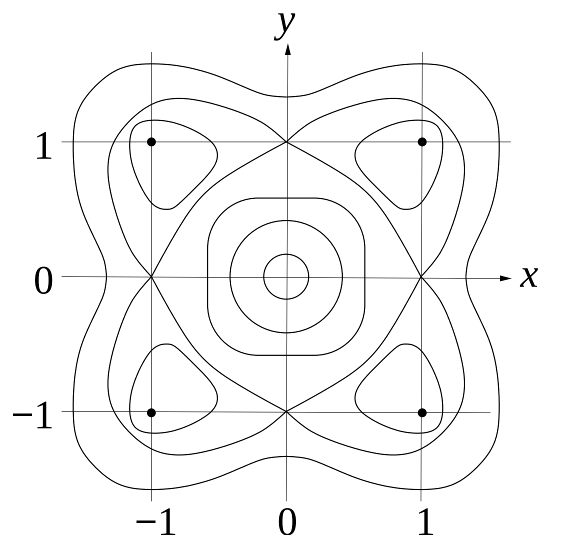
Dots: > 0**, White:** < 0**.**

(vi) 9 critical points, where $x, y = 0, 1, -1$**.**





Just add to obtain



o=max, •=min (×4), ×=saddle (×4)

Symmetry about $y = x$ and $y = -x$

All the critical points are non-degenerate, except in cases (iv), (v). In both these cases the origin is the only critical point, and is degenerate.