Question

Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that

(i) f is continuous everywhere (pay special attention to $x = 0$);

(ii) f is differentiable everywhere (likewise);

(iii) its derivative f' is *not* continuous at $x = 0$. [Thus f is differentiable everywhere but not C^1 in any region containing the origin.] Show that the function f^2 is C^1 but not C^2 in any region containing the origin.

Answer

(i) The function f is continuous at every $x \neq 0$ since it is obtained by multiplying and composing elementary continuous functions.

($\sin x$ is continuous everywhere as it is given as a convergent power series, or just check $|\sin(x+h) - \sin x| = |2\cos(x + \frac{h}{2}\sin\frac{h}{2}| < 2|\frac{h}{2}| = |h|$. So if $h \to 0$ then $\sin(x+h) \to \sin x$.)

At $x = 0$ we must verify explicitly

$$\begin{aligned} |f(h) - f(0)| &= |h^2 \sin(\frac{1}{h}) - 0| \\ &\leq |h^2| \end{aligned}$$

(as $|\sin\theta| \leq 1$ always)

so $f(h) \to f(0)$ as $h \to 0$.

(ii) The function f is differentiable at every $x \neq 0$ since it is obtained by multiplying and composing elementary differentiable forms.

Indeed, for $x \neq 0$ we have

$$\begin{aligned} f'(x) &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \end{aligned}$$

At $x = 0$ we observe explicitly

$$\begin{aligned} f(0+h) - f(0) &= h^2 \sin\frac{1}{h} \\ &= 0 + h \\ \text{and } |h\sin\left(\frac{1}{h}\right)| &\leq |h| \to 0 \text{ as } h \to 0 \end{aligned}$$

so f is differentiable at 0 with $f'(0) = 0$.

(iii) Looking at the above, we see $f'(x) \to f'(0)$ as $x \to 0$. (Indeed, $\lim_x \to 0 f'(x)$ doesn't exist). Thus f is not continuous at $x = 0$.

We have $\left.\begin{array}{ll} f^2 x & = x^4 \sin^2(\frac{1}{x}) \quad , x \neq 0 \\ & = 0 \qquad\qquad\quad , x = 0 \end{array}\right\}.$

We find

$$(f^2)'(x) = 4x^3 \sin\left(\frac{1}{x}\right) - 2x^2 \sin\left(\frac{1}{x}\right) \cos\left(\frac{1}{x}\right), \ x \neq 0$$

and (much as in (ii)) $(f^2)'(0) = 0$**.**

Since $|\sin\theta \cos\theta| \leq 1$ **for all** θ **we see**

$$(f^2)'(x) \to (f^2)'(0), \quad \textbf{as } x \to 0.$$

So $(f^2)'$ is continuous at 0 (and elsewhere), which says f^2 is c.

However, $(f^2)'$ is not c^1 in any region containing 0 ($\Rightarrow f^2$ is not c^2 there) as the 2^{nd} term is $-x^2 \sin(\frac{2}{x})$: compare (ii), (iii) above.