

QUESTION

(a) Sketch the region defined by the inequalities:

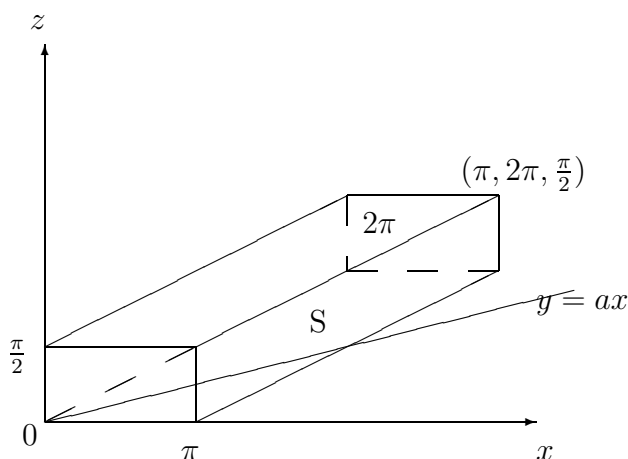
$$0 \leq x \leq \pi, \quad 0 \leq y \leq 2\pi, \quad 0 \leq z \leq \frac{\pi}{2}.$$

(b) If the region is occupied by a solid  $S$  with density at any point  $(x, y, z)$  given by the formula  $3x^2y \sin z$ , compute the total mass of the region  $S$  by evaluating an appropriate triple integral.

(c) The region  $S$  is divided by the plane  $y = ax$  (where  $a$  is a constant  $0 < a < 2$ ) into two regions: the region  $S_1$  contains the point  $(\pi, 0, 0)$  and the region  $S_2$  contains the point  $(0, 2\pi, 0)$ . Sketch the two regions  $S_1$  and  $S_2$ , and find the mass of  $S_1$  in terms of  $a$ .

(d) Using your answers to parts (b) and (c), find the mass of the upper part  $s_2$ , again in terms of  $a$ , and find the value of  $a$  for which the two regions have equal mass.

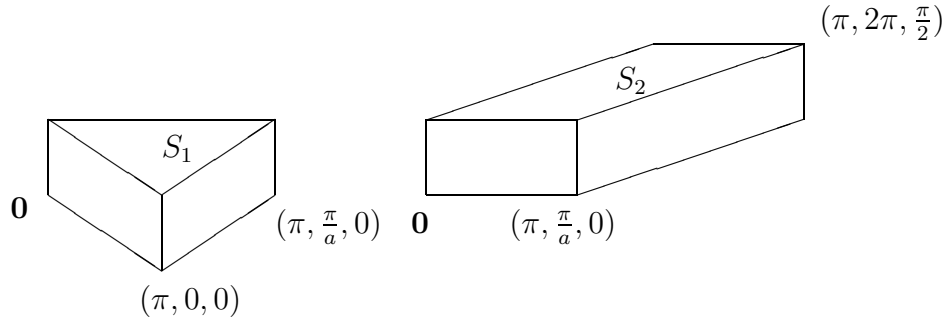
ANSWER



(a)

(b)

$$\begin{aligned} \int_0^\pi 3x^2 dx \int_0^{2\pi} y dy \int_0^{\frac{\pi}{2}} \sin z dz &= [x^3]_0^\pi \left[ \frac{y^2}{2} \right]_0^{2\pi} [-\cos z]_0^{\frac{\pi}{2}} \\ &= \pi^3 \frac{4\pi^2}{2} = 2\pi^5 \end{aligned}$$



(c)

$$\begin{aligned}
 \text{Mass} S_1 &= \int_0^{\frac{\pi}{2}} \int_{x=0}^{\pi} \int_{y=0}^{ax} 3x^2 y \sin z \, dy dx dz \\
 &= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \left[ 3x^2 \sin z \frac{y^2}{2} \right]_0^{ax} dx dz \\
 &= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \frac{3a^2}{2} x^4 \sin z \, dx dz \\
 &= \int_0^{\frac{\pi}{2}} \left[ \frac{3a^2}{2} \frac{x^5}{5} \sin z \right]_0^{\pi} dz \\
 &= \int_0^{\frac{\pi}{2}} \frac{3}{10} a^2 \pi^5 \sin z \, dz \\
 &= \left[ \frac{3}{10} a^2 \pi^5 (-\cos z) \right]_0^{\frac{\pi}{2}} \\
 &= \frac{3}{10} a^2 \pi^5
 \end{aligned}$$

(d)

$$\begin{aligned}
 \text{mass} S_2 &= \text{mass} S - \text{mass} S_1 \\
 &= 2\pi^5 - \frac{3}{10} a^2 \pi^5 = \left( \frac{20 - 3a^2}{10} \right) \pi
 \end{aligned}$$

$$\text{Mass } S_1 = \text{Mass } S_2 \Leftrightarrow \frac{3}{10} a^2 = \frac{20 - 3a^2}{10} \Leftrightarrow 6a^2 = 20 \Leftrightarrow a^2 = \frac{10}{3} \Leftrightarrow a = \sqrt{\frac{10}{3}}$$