QUESTION

The matrices A and B are defined to be

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \\ 4 & 1 & 1 \end{pmatrix}.$$

(a) Compute the determinants of each of the matrices $A, B, a+B, AB$.

(b) Find the eigenvalues for the matrix A (you are NOT required to find the eigenvectors). [Hint: Use row reduction to simplify the computation.]

(c) The matrix B has three distinct eigenvalues. Compute them and find, and normalise the corresponding eigenvectors. Verify that these vectors form an orthonormal system.

ANSWER

(a)

$$\det(A) = 4\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 4.15 - 4 + 1 + 1 - 4 = 54$$

$\det(B) = -1\det(A) = -54$ since B is obtained by switching rows 1 and 3.

$$\det(A+B) = \begin{vmatrix} 5 & 2 & 5 \\ 2 & 8 & 2 \\ 5 & 2 & 5 \end{vmatrix} = 0 \text{ since row 1=row 3}$$

$\det(AB) = \det(A)\det(B) = -(54)^2 = -2916$

(b) The eigenvalues satisfy $|A - \lambda I| = 0$

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} &= \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \\
&= (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \\
&= (6-\lambda)\begin{vmatrix} 0 & \lambda-3 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \\
&= (6-\lambda)(3-\lambda)(3-\lambda)
\end{aligned}$$

$\lambda = 3, 3, 6.$

(c)

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 & 4 \\ 1 & 4-\lambda & 1 \\ 4 & 1 & 1-\lambda \end{vmatrix} \\
&= (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 4 & 1 & 1-\lambda \end{vmatrix} \\
&= (6-\lambda)\begin{vmatrix} 0 & \lambda-3 & 0 \\ 1 & 4-\lambda & 1 \\ 4 & 1 & 1-\lambda \end{vmatrix} \\
&= (6-\lambda)(\lambda-3)(-3-\lambda)
\end{aligned}$$

so $\lambda = 6, 3, -3.$

For the eigenvectors,

$\lambda = -3$

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x + y + 4z \\ x + 7y + z = 0 \end{aligned}\right\} \begin{aligned} 44 + y + 4z = 0 \\ 4x + 28y + 4z = 0 \end{aligned}$$

$27y = 0$ $y = 0,\ x = -z$ so putting $x = 1$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ which normalises to

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{v}_1$

$\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -2x + y + 4z = 0 \\ x + y + z = 0 \end{aligned}\right\} \left.\begin{aligned} -2x + y + 4z = 0 \\ -2x - 2y - 2z = 0 \end{aligned}\right\} 3y + 6z = 0$$

so putting $z = 1, y = -2 \Rightarrow x = 1$, $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ which normalises to $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \mathbf{v}_2$

$\lambda = 6$

$$\begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} -5x + y + 4z = 0 \\ x - 2y + z \end{array}\right\} \quad \left.\begin{array}{r} -5x + y + 4z \\ -5x + 10y - 5z \end{array}\right\} 0x - 9y + 9z$$

$y = z$ so putting $y = 1, z = 1 \Rightarrow x = 1$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which normalises to $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{v}_3$

$$\begin{aligned} \mathbf{v}_1.\mathbf{v}_2 &= \frac{1}{\sqrt{12}}(1.1 + 0 + 1. - 1) = 0 \\ \mathbf{v}_1\mathbf{v}_3 &= \frac{1}{\sqrt{6}}(1.1 + 0.1 + -1.1) = 0 \\ \mathbf{v}_2.\mathbf{v}_3 &= \frac{1}{\sqrt{18}}(1.1 - 2.1 + 1.1) \end{aligned}$$

So the vectors do form an orthonormal system.