QUESTION

(a) Find the solution to the following differential equation:

$$(x^3-7)\frac{dy}{dx}+3x^2y^2=0,$$

subject to the initial condition $y=1$ when $x=2$.

(b) Find the solution to the following differential equation

$$x\frac{dy}{dx}+2y=\sin(x),$$

subject to the initial condition $y=\frac{4}{\pi^2}$ when $x=\frac{\pi}{2}$.

(c) Fins the solution to the following differential equation given the conditions:

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0,$$

subject to the initial conditions
$y(0)=1,\ \frac{dy}{dx}(0)=5.$

ANSWER

(a) Rearranging we get $\dfrac{dy}{dx}=\dfrac{-3x^2y^2}{(x^3-7)}$, which is separable with $\dfrac{1}{y^2}\dfrac{dy}{dx}=\dfrac{-3x^2}{(x^3-7)}$ giving $\dfrac{-1}{y}=\displaystyle\int\frac{-3x^2}{x^3-7}\,dx$. Now put $u=x^3-7$ to get $\dfrac{du}{dx}=3x^2$ so that $\displaystyle\int\frac{-3x^2}{x^3-7}\,dx=\int\frac{-\frac{du}{dx}}{u}\,dx=\int\frac{-1}{u}\,du=-\log|u|+k-\log|x^3-7|+k$. Hence $y=\dfrac{1}{\log|x^3-7|-k}$. Putting $y=1$ when $x=2$ gives $1=\dfrac{1}{-k}$ so $k=-1$ and $y=\dfrac{1}{\log|x^3-7|+1}$.

(b) Assuming $x\neq 0$ simplify to get $\dfrac{dy}{dx}+\dfrac{2}{x}y=\dfrac{\sin(x)}{x}$ which is first order linear with integrating factor $e^{\int\frac{2}{x}\,dx}=x^2$. The equation becomes $x^2y=\displaystyle\int x\sin x\,dx$. Integrate by parts with $u=x$ and $\dfrac{dv}{dx}=\sin x$ to get

$x^2y = [-x\cos x] + \sin x + k$, so $y = \dfrac{-\cos x}{x} + \dfrac{\sin x}{x^2} + \dfrac{k}{x^2}$. Putting $y = \dfrac{4}{\pi^2}$ when $x = \dfrac{\pi}{2}$ we get $k = 0$ so $y = \dfrac{-\cos x}{x} + \dfrac{\sin x}{x^2}$.

(c) Second order homogeneous linear with auxilliary equation $\lambda^2 - 2\lambda + 5 - 0$. Roots $1 \pm 2i$ so general solution has the form $y = e^x[A\cos 2x + B\sin 2x]$. Then

$$\begin{aligned} \frac{dy}{dx} &= e^x[-2A\sin 2x + 2B\cos 2x] + e^x[A\cos 2x + B\sin 2x] \\ &= e^x[(2B + A)\cos x + (B - 2A)\sin x] \end{aligned}$$

Putting $y(0) = 1$, and $\dfrac{dy}{dx}(0) = 5$ we get $A + 2B = 5$ $A = 1 \Rightarrow B = 2$ so $y = e^x[\cos 2x + 2\sin 2x]$.