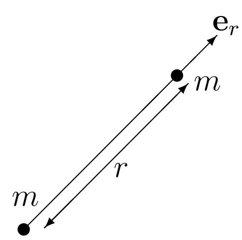
Question

A particle of mass m moves under the influence of the gravitational force of a particle of mass M fixed at the origin. Find its equations of motion in spherical polar coordinates. Show that $r^2 \sin^2\theta\dot{\phi}$ is constant during the motion.

Answer

$$m\ddot{\mathbf{r}} = -\frac{GMm}{r^2}\mathbf{e}_r$$



Therefore using $\mathbf{a}$ in spherical polar coordinates

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\mathbf{r}_r + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2\right]\mathbf{e}_\theta \\ &\quad + \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi})\mathbf{e}_\phi\end{aligned}$$

we find on equating components that

$$\begin{aligned}(\mathbf{e}_r) \qquad \ddot{r} - r\dot{\theta}^2 - r\sin\theta\dot{\phi}^2 &= -\frac{GM}{r^2} \\ (\mathbf{e}_\theta) \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ (\mathbf{e}_\phi) \qquad \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}) &= 0\end{aligned}$$

By integrating the $\mathbf{e}_\phi$ equation we obtain $r^2\sin^2\theta\dot{\phi} = \text{constant}$.