QUESTION

In a sweepstake there are N tickets ($N > 5$), and five prizes are to be won. The prizes are allocated by drawing five tickets from a hat one after the other. Each winning ticket is replaced in the hat before the next ticket is drawn, so that a ticket may win more than one prize. The number of prizes won by a competitor who has brought two tickets is a random variable R. State the value of $E(R)$ and show that $\mathrm{var}(R) = \frac{10(N-2)}{N^2}$.

The rules are changed so that winning tickets are no longer replaced. The number of prizes won by a competitor with two tickets is S.

Find $P(S = 0), P(S = 1), P(S = 2)$.

Show that $E(S) = E(R)$ and $\mathrm{var}(S) = \frac{(N-5)\mathrm{var}(R)}{(N-1)}$.

ANSWER

If he buys two tickets, probability of winning on each draw is $\frac{2}{N}$. $R \sim B(5, \frac{2}{N})$ since the tickets are replaced.

$\mu = 5 \times \frac{2}{N} = \frac{10}{N}, \sigma^2 = 5 \times \frac{2}{N}(1 - \frac{2}{N}) = \frac{10(N-2)}{N^2}$ Without replacement the distribution is hypergeometric.

$$
\begin{aligned}
P(S = 0) &= \frac{\begin{pmatrix} N-2 \\ 5 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}}{\begin{pmatrix} N \\ 5 \end{pmatrix}} \\
&= \frac{(N-2)(N-3)(N-4)(N-5)(N-6)}{N(N-1)(N-2)(N-3)(N-4)} \\
&= \frac{(N-5)(N-6)}{N(N-1)}
\end{aligned}
$$

$$
\begin{aligned}
P(S = 1) &= \frac{\begin{pmatrix} N-2 \\ 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}{\begin{pmatrix} N \\ 5 \end{pmatrix}} \\
&= \frac{(N-2)(N-3)(N-4)(N-5) \times 2 \times 5}{4!N(N-1)(N-2)(N-3)(N-4)} \\
&= \frac{10(N-5)}{N(N-1)}
\end{aligned}
$$

$$
\begin{aligned}
P(S=2) &= \frac{\binom{N-2}{3}\binom{2}{2}}{\binom{N}{5}} \\
&= \frac{(N-2)(N-3)(N-4)5!)}{3!N(N-1)(N-2)(N-3)(N-4)} \\
&= \frac{20}{N(N-1)}
\end{aligned}
$$

Check that these add to give 1.

$$
E(S) = \frac{10(N-5)}{N(N-1)} + \frac{40}{N(N-1)} = \frac{10}{N}
$$

$$
\begin{aligned}
\mathrm{Var}(S) &= \frac{10(N-5)}{N(N-1)} + \frac{80}{N(n-1)} - \frac{100}{N^2} \\
&= \frac{10(N^2-7N+10)}{N^2(N-1)} \\
&= \frac{10(N-2)(N-5)}{N^2(N-1)} \\
&= \frac{(N-5)}{(N-1)}\mathrm{Var}(R)
\end{aligned}
$$