Question

Write down two explicit Möbius transformations taking the disc $D = \{z \in \mathbf{C} \mid |z-2| < 1\}$ to the disc $E = \{z \in \mathbf{C} \mid \mathrm{Im}(z) < 0\}$.

Answer

Start by taking $\{|z-2| = 1\}$ to $\bar{\mathbf{R}}$:
take 3 points on $A = \{|z-2| = 1\}$, such as $z_1 = 1, z_2 = 3, z_3 = 2+i$ and then take $m \in \text{Möb}^+$ satisfying $m(z_1) = 0, m(z_2) = \infty, m(z_3) = 1$:

$$m(z) = \frac{z-1}{z-3} \cdot \frac{2+i-3}{2+i-1} = \frac{z-1}{z-3} \cdot \frac{-1+i}{1+i}$$

Since $m(A) = \bar{\mathbf{R}}$, either $m(D) = \mathbf{H}$ or $m(D) = E = \{\mathrm{Re(z)} < 0\}$. Test by checking $m(2)$ since $2 \in D$:

$$\begin{aligned} m(z) &= \frac{2-1}{2-3} \cdot \frac{-1+i}{1+i} \\ &= \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} \\ &= \frac{-2i}{2} = -i \in E \end{aligned}$$

and so $m(D) = E$ as desired.
A second one is $\underline{p(z) = m(z) + 1}$.