Question
Find the solutions of the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + (1-a)y = 0$$

which satify the conditions $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$ for three cases:

(i) a is a positive real number, so we can put $a = k^2$ with $k > 0$;

(ii) a is a negative real number, so we can put $a = -p^2$ with $p > 0$;

(iii) $a = 0$

Answer
Auxiliary equation: $\lambda^2 + 2\lambda + (1-a) = 0$
Roots: $\lambda_1, \lambda_2 = \frac{-2\pm\sqrt{4-4(1-a)}}{2} = -1 \pm \sqrt{a}$

case (i)

$$\begin{aligned}
0 < a &= k^2 \text{ this gives two distinict real roots} \\
\lambda_1, \lambda_2 &= -1 \pm k \\
\text{General Solution : } y &= Ae^{(k-1)x} + Be^{-(k+1)x} \\
\frac{dy}{dx} &= (k-1)Ae^{(k-1)x} - (k+1)Be^{-(k+1)x} \\
\text{Boundary conditions : } 0 &= A + B \\
1 &= (k-1)A - (k+1)B \\
\Rightarrow A &= \frac{1}{2k} \text{ and } B = -\frac{1}{[}2k \\
\text{Particular solution : } y &= \frac{e^{-x}}{k}\left\{\frac{e^{kx} - e^{-kx}}{2}\right\} \\
&= \frac{e^{-x}}{k}\sinh(kx)
\end{aligned}$$

case (ii)

$$\begin{aligned}
0 > a &= -p^2 \text{this gives a pair of complex conjugate roots} \\
\lambda_1, \lambda_2 &= -1 \pm \sqrt{-p^2} \\
\text{General Solution : } y &= e_{-x}(A\cos(px) + B\sin(px)) \\
\frac{dy}{dx} &= e^{-x}((pB - A)\cos(px) - (pA + B)\sin px) \\
\text{Boundary conditions : } 0 &= A
\end{aligned}$$

$$
\begin{aligned}
1 &= pB - A \\
\Rightarrow A &= 0 \text{ and } B = -\frac{1}{p} \\
\text{Particular solution : } y &= \frac{e^{-x}}{p}\sin(px)
\end{aligned}
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case (iii)

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\begin{aligned}
a &= 0 \text{ this gives two repeated real roots} \\
\lambda_1, \lambda_2 &= -1 \\
\text{General Solution : } y &= (A + Bx)e^{-x} \\
\frac{dy}{dx} &= (-A + B - Bx)e^{-x} \\
\text{Boundary conditions : } 0 &= A \\
1 &= -A + B \\
\Rightarrow A &= 0 \text{ and } B = 1 \\
\text{Particular solution : } y &= xe^{-x}
\end{aligned}
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