Question

Using Lagrange's method, obtain the general solution of $uu_x + u_y = 1$. If $u = 0$ on $y^2 = 2x$ obtain the solution and state the region in which it is valid.

Answer

Lagrange gives

$\frac{dx}{d\xi} = u,\ \frac{dy}{d\xi} = 1,\ \frac{du}{d\xi} = 1$

$\Rightarrow u = \xi + a,\ y = \xi + b \quad (1)$

$\Rightarrow \frac{dx}{d\xi} = \xi + a$

$\Rightarrow x = \frac{1}{2}\xi^2 + a\xi + c \quad (2)$

where $a,\ b,\ c$ are constants

Therefore we have ((1)): $u - y = const \quad (3)$

and ((2)): $x = \frac{1}{2}(u-a)^2 + a(u-a) - +c$

$\Rightarrow x = \frac{1}{2}u^2 + const$

$\Rightarrow \left(x - \frac{1}{2}u^2\right) = const \quad (4)$

Therefore (3) and (4) give general solution

$f\left(u - y,\ x - \frac{1}{2}u^2\right) = const$

Write this as $u - y = g\left(x - \frac{1}{2}u^2\right)$ say.

Boundary condition then gives $-\sqrt{2x} = g(x)$, so

$g\left(x - \frac{1}{2}u^2\right) = -\sqrt{2x - u^2}$.

Thus $u - y = -\sqrt{2x - u^2} \longrightarrow u = \frac{y}{2} \pm \left(x - \frac{y^2}{4}\right)^{\frac{1}{2}}$.

Since $u = 0$ on $y^2 = 2x$ we need negative root, so

$$\underline{u = \frac{1}{2}y - \left(x - \frac{u^2}{4}\right)}$$

No solution for $x < \frac{y^2}{4}$.