Question

Find all the solutions of $u_t + cu_x = 0$ where c is a constant which satisfy $u_t(x,0) + ku(x,0) = \phi(x)$, where k is a constant and ϕ is a given function.

Answer

The general solution of $u_t + cu_x = 0$ is given by:

$$\frac{dt}{d\xi} = 1,\ \frac{dx}{d\xi} = c,\ \frac{du}{d\xi} = 0$$

$$\frac{dt}{dx} = \overset{!}{-}c,\ u = const$$

$$\Rightarrow \frac{dx}{dt} = c,\ u = const$$

$$\Rightarrow x = \alpha + ct,\ u = \beta \qquad (alpha,\ beta = const.)$$

Therefore $\left\{ \begin{array}{rcl} x - ct & = & const \\ u & = & const \end{array} \right.$

$$\Rightarrow \underline{u = f(x - ct)}$$

Therefore boundary condition is satisfied by:

$$-cf'(x) + kf(x) = \phi(x)$$

(1st order linear: solve with integrating factor $e^{\frac{-kx}{c}}$)

This ODE can be solved to give:

$$f'(x) - \frac{k}{c}f9x) = \frac{1}{c}\phi(x)$$

$$\Rightarrow e^{\frac{-kx}{c}}f'(x) - \frac{k}{c}e^{\frac{-kx}{c}}f(x) = \frac{e^{\frac{-kx}{c}}}{c}\phi(x)$$

$$\Rightarrow \frac{d}{dx}\left[e^{\frac{-kx}{c}}f(x)\right] = \frac{e^{\frac{-kx}{c}}}{c}\phi(x)$$

$$\Rightarrow e^{\frac{-kx}{c}}f(x) = \frac{1}{c}\int_a^x d\eta\phi(\eta)e^{\frac{-kx}{c}}$$

$$\Rightarrow \underline{f(x) = \frac{e^{\frac{-kx}{c}}}{c}\int_a^x d\eta\phi(\eta)e^{\frac{-kx}{c}}}$$

where a is an arbitrary constant.

So the specific solution is:

$$u(x,t) = \frac{e^{\frac{k(x-ct)}{c}}}{c}\int_a^{x-ct} e^{\frac{-k\eta}{c}}\phi(\eta)d\eta$$

where a is an arbitrary constant.