Question

Find the first order partial differential equations for which the following are general solutions and describe them as linear, homogeneous,etc. as appropriate (in each case f is an arbitrary real function)

(a) $u = xf(x^2 + y^2)$

(b) $u = xy + (x - y)f(x + y)$

(c) $u = x + f(uy)$

(d) $u = f\left(\dfrac{xy}{u}\right)$

Answer

(a)
$$\begin{array}{rcl} u_x &=& f(x^2+y^2) \quad + \quad 2x^2 f'(x^2+y^2) \\ u_y &=& 2yxf'(x^2+y^2) \end{array}$$

Therefore

$$\begin{array}{rl} xyu_x = & xyf + 2x^3yf' \quad + \\ -x^2u_y = & -2x^3yf' \\ \hline \end{array}$$

$xyu_x - x^2u_y = xyf$

or $xyu_x - x^2u_y = yu$, linear and homogeneous.

(b) $u_x = y + f(x + y) + 9x - y)f'(x + y)$

$u_y = x - f(x + y) + (x - y)f'(x + y)$

Therefore $(x - y)(u_x - u_y) = 2u - (x^2 + y^2)$, linear <u>non</u>-homogeneous

(c)
$$\begin{array}{rcl} u_x &=& 1 + f'(uy)u_x \Rightarrow u_x(1 - f'(uy) = 1 \\ u_y &=& f'(uy)(yu_y + u) \end{array}$$

$$\begin{array}{rcl} uu_x &=& u + uu_xf' \\ -yu_y &=& - f'(yu_y + u) \end{array}$$

$\Rightarrow \underline{uu_x - yu_y = u}$, quasi-linear

(d) $xu_x - yu_y = 0$ linear, homogeneous