Question
A pdf is defined by

$$f(x,y) = \begin{cases} Cy^2 & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of C.

(b) Find $P(X+Y>2),\ P(Y<\frac{1}{2}),\ P(X\le 1),\ P(X=3Y)$.

(c) Are X and Y independent?

(d) Are the events $\{X<1\}$ and $\{Y \ge \frac{1}{2}\}$ independent?

Answer

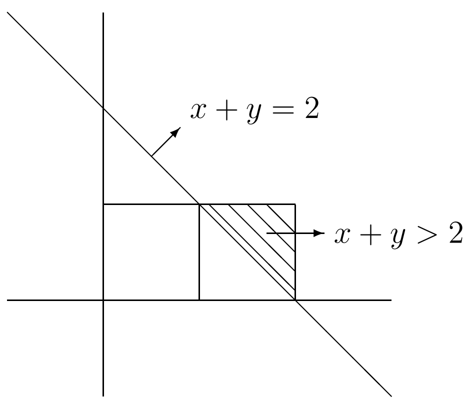
(a) $f(x,y) = Cy^2, \quad 0 \le x \le 2.$

$$\int_0^2 \int_0^1 y^2\,dy\,dx = \int_0^2 dx\,\frac{y^3}{3}\bigg|_0^1 = \frac{2}{3} \Rightarrow C = \frac{3}{2}$$

(b) $f_X(x) = \frac{1}{2}, \quad 0<x<2$

$f_Y(y) = 3y^2, \quad 0<y<1$

X and Y are independent.



$$\begin{aligned} P(X+Y>2) &= \frac{3}{2}\int_1^2 dx \int_{2-x}^1 y^2\,dy \\ &= \frac{3}{2}\int_1^2 dx\,\frac{y^2}{3}\bigg|_{2-x}^1 \end{aligned}$$

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\begin{aligned}
&= \frac{1}{2}\int_1^2 \left\{1-(2-x)^3\right\}\, dx \\
&= \frac{3}{8}
\end{aligned}
$$

$P\left\{Y < \dfrac{1}{2}\right\} = \displaystyle\int_0^{\frac{1}{2}} 3y^2\, dy = \dfrac{1}{8}$

$P\left\{X \le 1\right\} = \displaystyle\int_0^1 \dfrac{1}{2}\, dx = \dfrac{1}{2}$

$P(X = 3Y) = 0$ because the distribution of X and Y is continuous.

(c) $P\left(X \le 1, Y \ge \dfrac{1}{2}\right) = \dfrac{3}{2}\displaystyle\int_0^1 dx \int_{\frac{1}{2}}^1 y^2\, dy = \dfrac{7}{16}$

$P(X \le 1)\cdot P(Y \ge \frac{1}{2}) = \dfrac{1}{2}\cdot\dfrac{7}{8} = \dfrac{7}{16}$

Hence the events are independent.

(d) Also follows from the general result that: if X and Y are independent then

$$P\left\{X \in A, Y \in B\right\} = P(X \in A, Y \in B).$$