Question

Suppose that X and Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

By using a suitable transformation, show that the pdf of $U = \frac{(X+Y)}{2}$ is given by

$$f_U(u) = 4ue^{-2u},\ u > 0.$$

Answer

Let $Z = \dfrac{X_1 + X_2}{2}$ and $W = X_2$

$\Rightarrow x_1 = 2z + w; \quad x_2 = w$

$\Rightarrow 0 < x_1 < \infty; \quad 0 < 2z - w < \infty$

$\begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial w_2}{\partial z} & \frac{\partial x_2}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$

Therefore the joint pdf of (Z,W) is

$f_{Z,W}(z,w) = e^{-2z}.2, \quad 0 < 2z - w < \infty$

Therefore $f_Z(z) = 2\int_0^{2z} e^{-2z}\, dw = 4ze^{-2z}, \quad z > 0$