Question

Suppose that X and Y are independent standard normal random variables.

(a) Prove that $Z = aX + bY$ is normally distributed where a and b are given constants such that both are not equal to zero at the same time. Give the mean and variance of Z.

(b) Find $P(X - Y > 0)$ and $P(X - Y > 1)$.

(c) Derive the distribution of $Z = X^2 + Y^2$. Hence find $P(X^2 + Y^2 \leq 1)$.

Answer

(a) $M_X(t) = E(e^{tX}) = e^{\frac{t^2}{2}}$

$M_Y(t) = E(e^{tY}) = e^{\frac{t^2}{2}}$

$$\begin{aligned} M_Z(t) &= E(e^{tZ}) \\ &= E\left\{e^{t(aX+bY)}\right\} \\ &= E(e^{atX}e^{btY}) \\ &= M_X(at)M_Y(bt) \text{ since } X \text{ and } Y \text{ are independent} \\ &= e^{\frac{1}{2}(a^2+b^2)t^2} \end{aligned}$$

The above is the mgf of a normal r.v. with mean 0 and variance a^2+b^2. By using the uniqueness theorem of the mgf $Z \sim N(0, a^2+b^2)$.

(b) $Z = X - Y \sim N(0, 2)$

$P(Z > 0) = \frac{1}{2}$

$P(Z > 0) = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right) = 1 - 0.76 = 0.24$

(c) $Z = X^2 + Y^2 \sim \chi^2$ with 2 degrees of freedom by using the mgf technique $\star$

$$\begin{aligned} P(Z \leq 1) &= \int_0^1 f(z)\,dz = \int_0^1 \frac{1}{\Gamma\left(\frac{2}{2}\right)2^{\frac{2}{2}}} z^{\frac{2}{2}-1}e^{-\frac{1}{2}z}\,dz \\ &= \int_0^1 \frac{1}{2}e^{-\frac{1}{2}z}\,dz = 1 - e^{-\frac{1}{2}} \end{aligned}$$

$\star M_Z(t) = E\left(e^{tX^2+tY^2}\right) = \left(\frac{1}{1-2t}\right)^{\frac{2}{2}}$ if $t < \frac{1}{2}$