QUESTION Show that every 2×2 matrix A for which $A^2 = 0$ has det$A$=0 and tr$A$=0.

ANSWER

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{bmatrix}$$

If $b \neq 0$ and $c \neq 0$ then $a + d =$tr$A = 0$ and $a^2 + bc = 0$. Hence det$A = ad - bc = -a^2 - bc = 0$;or use $\det(A^2) = (\det A)^2$. [The cases where either b or c=0 need to be considered separately.]

The general matrix with $A^2 = 0$ (and $b \neq 0$) can be written

$$\begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix}$$