Question

a) State Liouville's Theorem, and use it to prove the Fundamental Theorem of Algebra.

b) Locate the zeros and singularities of the function

$$\frac{(z^2-4)\cos\left(\frac{1}{z}\right)}{z^2+z-6}$$

Classify the singularities, and determine the behaviour of the function at infinity.

Answer

a) Liouville's Theorem states that if $f(z)$ is analytic for all z, and is bounded then $f(z)$ is constant.

Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \quad n \geq 1$ be a polynomial (with complex coefficients). Suppose $P(z) \neq 0$, for all $z \in \mathbf{C}$.

Let $f(z) = \frac{1}{P(z)}$, and so $f(z)$ is analytic for all $z \in \mathbf{C}$.

Now $\frac{P(z)}{z^n} \to 1$ as $|z| \to \infty$. Thus $\exists R,\ |z| > R \Rightarrow |f(z)| < 1$. Now since $f(z)$ is continuous, it is bounded in $|z| \leq R$. Thus $f(z)$ is bounded and so constant. So $f(z) \equiv f(0) = \frac{1}{P(o)} \neq 0$.

This contradicts $f(z) \to 0$ as $|z| \to \infty$. Hence $P(z)$ must have a zero.

b) $f(z) = \dfrac{(z+2)(z-2)\cos\left(\frac{1}{z}\right)}{(z-2)(z+3)}$

So we have

i) a removable singularity at $z = 2$.

ii) a simple pole at $z = -3$.

iii) an essential singularity at $z = 0$.

iv) a zero at $z = -2$.

v) zeros where $\frac{1}{z} = (2n+1)\frac{\pi}{2}$, $\quad n \in \mathbf{Z}$

$$f\left(\frac{1}{z}\right) = \frac{\left(\frac{1}{z}+2\right)\left(\frac{1}{z}-2\right)\cos z}{\left(\frac{1}{z}-2\right)\left(\frac{1}{z}+3\right)} \to 1 \text{ as } z \to 0$$

Hence f is analytic at infinity.