Question

Show that the element of arc-length $\frac{c}{\mathrm{Im}(z)}|\mathrm{dz}|$ on $\mathbf{H}$ is invariant under $B(z) = -\overline{z}$.

Answer

Let $f : [a, b] \longrightarrow \mathbf{H}$, $f(t) = x(t) + iy(t)$ be a piecewise differentiable path. Then,

$$\mathrm{length(f)} = \int_{\mathrm{f}} \frac{\mathrm{c}}{\mathrm{Im(z)}}\,|\mathrm{dz}|$$

$B \circ f(t) = -\overline{f(t)} = -x(t) + iy(t)$. Thus, $\mathrm{Im}(B \circ f(t)) = \mathrm{Im(f(t))}$ and $|(B \circ f)'(t)| = |f'(t)|$, and so

$$\begin{aligned} \mathrm{length(B \circ f)} &= \int_{B \circ f} \frac{c}{\mathrm{Im(z)}}|\,dz| \\ &= \int_a^b \frac{c}{\mathrm{Im(B \circ f(t))}}|(B \circ f)'(t)|\,dt \\ &= \int_a^b \frac{c}{\mathrm{Im(f(t))}}|f'(t)|\,dt \\ &= \text{length(f) as desired.} \end{aligned}$$