Question

Determine whether the following improper integrals converge or diverge, and evaluate those which converge.

1. $\int_0^4 \mathrm{d}x/x^{3/2}$;
2. $\int_1^\infty \mathrm{d}x/(x+1)$;
3. $\int_5^\infty \mathrm{d}x/(x-1)^{3/2}$;
4. $\int_0^9 \mathrm{d}x/(9-x)^{3/2}$;
5. $\int_{-\infty}^{-2} \mathrm{d}x/(x+1)^3$;
6. $\int_{-1}^{8} \mathrm{d}x/x^{1/3}$;
7. $\int_2^\infty \mathrm{d}x/(x-1)^{1/3}$;
8. $\int_{-\infty}^{\infty} x\mathrm{d}x/(x^2+4)$;
9. $\int_0^1 e^{\sqrt{x}}\mathrm{d}x/\sqrt{x}$;
10. $\int_1^\infty \mathrm{d}x/x\ln(x)$;

Answer

1. this is an improper integral because $1/x^{3/2}$ is continuous on $(0, 4]$ and $\lim_{x\to 0+} 1/x^{3/2} = \infty$. So, we evaluate:

$$\begin{aligned} \int_0^4 \frac{1}{x^{3/2}}\,\mathrm{d}x &= \lim_{c\to 0+} \int_c^4 \frac{1}{x^{3/2}}\,\mathrm{d}x \\ &= \lim_{c\to 0+} \int_c^4 x^{-3/2}\,\mathrm{d}x \\ &= \lim_{c\to 0+} \left(-\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{c}}\right) \\ &= -1 + 2\lim_{c\to 0+} \frac{1}{\sqrt{c}} = \infty, \end{aligned}$$

   and so this improper integral **diverges**.

2. this is an improper integral because the interval of integration is $[1, \infty)$, which is not a closed interval. So, we evaluate:

$$\begin{aligned} \int_1^\infty \frac{1}{x+1}\,\mathrm{d}x &= \lim_{M\to\infty} \int_1^M \frac{1}{x+1}\,\mathrm{d}x \\ &= \lim_{M\to\infty} \left[\ln(M+1) - \ln\left(\frac{1}{2}\right)\right] = \infty, \end{aligned}$$

   and so this improper integral **diverges**.

3. this is an improper integral, as the interval of integration is $[5, \infty)$, which is not a closed interval. So, we evaluate:

$$\begin{aligned}\int_5^\infty \frac{1}{(x-1)^{3/2}}\,\mathrm{d}x &= \lim_{M\to\infty}\int_5^M \frac{1}{(x-1)^{3/2}}\,\mathrm{d}x \\ &= \lim_{M\to\infty}\int_5^M (x-1)^{-3/2}\,\mathrm{d}x \\ &= \lim_{M\to\infty}\left[-\frac{2}{\sqrt{M-1}}+1\right] = 1,\end{aligned}$$

and so this improper integral **converges to** 1.

4. this is an improper integral because $1/(9-x)^{3/2}$ is continuous on $[0, 9)$ and $\lim_{x\to 9-} 1/(9-x)^{3/2} = \infty$. So, we evaluate:

$$\begin{aligned}\int_0^9 \frac{1}{(9-x)^{3/2}}\,\mathrm{d}x &= \lim_{c\to 9-}\int_0^c \frac{1}{(9-x)^{3/2}}\,\mathrm{d}x \\ &= \lim_{c\to 9-}\int_0^c (9-x)^{-3/2}\,\mathrm{d}x \\ &= \lim_{c\to 9-}\left[-\frac{2}{3}+\frac{2}{\sqrt{9-c}}\right] = \infty,\end{aligned}$$

and so this improper integral **diverges**.

5. this is an improper integral, since the interval of integration is $(-\infty, -2]$ and so is not a closed interval. So, we evaluate:

$$\begin{aligned}\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,\mathrm{d}x &= \lim_{M\to-\infty}\int_M^{-2} \frac{1}{(x+1)^3}\,\mathrm{d}x \\ &= \lim_{M\to-\infty}\left[-\frac{1}{2}\frac{1}{(-2+1)^2}+\frac{1}{2}\frac{1}{(M+1)^2}\right] = -\frac{1}{2},\end{aligned}$$

and so this improper integral **converges to** $-\frac{1}{2}$.

6. this is an improper integral, since the integrand is not continuous on $[-1, 8]$ as it has a discontinuity at 0. Hence, we can break it up as the sum of two improper integrals:

$$\int_{-1}^8 \mathrm{d}x/x^{1/3} = \int_{-1}^0 \mathrm{d}x/x^{1/3} + \int_0^8 \mathrm{d}x/x^{1/3},$$

and we have that $\int_{-1}^8 \mathrm{d}x/x^{1/3}$ converges if both $\int_{-1}^0 \mathrm{d}x/x^{1/3}$ and $\int_0^8 \mathrm{d}x/x^{1/3}$ converge. So, we evaluate:

$$\int_{-1}^0 \frac{1}{x^{1/3}}\mathrm{d}x = \lim_{c\to 0-}\int_{-1}^c \frac{1}{x^{1/3}}\mathrm{d}x$$

$$
\begin{aligned}
&= \lim_{c\to 0-}\int_{-1}^{c} x^{-1/3}\mathrm{d}x \\
&= \lim_{c\to 0-}\left[\frac{3}{2}c^{2/3}-\frac{3}{2}\right] = -\frac{3}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^8 \frac{1}{x^{1/3}}\mathrm{d}x &= \lim_{c\to 0+}\int_c^8 \frac{1}{x^{1/3}}\mathrm{d}x \\
&= \lim_{c\to 0+}\int_c^8 x^{-1/3}\mathrm{d}x \\
&= \lim_{c\to 0+}\left[\frac{3}{2}8^{2/3}-\frac{3}{2}c^{2/3}\right] = 6.
\end{aligned}
$$

Since both these improper integrals converge, we see that the original improper integral $\int_{-1}^{8}\mathrm{d}x/x^{1/3}$ **converges to** $\frac{9}{2}$.

7. this is an improper integral, since the interval of integration is $[2,\infty)$ and hence is not a closed interval. So, we evaluate:

$$
\begin{aligned}
\int_2^\infty \frac{1}{(x-1)^{1/3}}\,\mathrm{d}x &= \lim_{M\to\infty}\int_2^M \frac{1}{(x-1)^{1/3}}\,\mathrm{d}x \\
&= \lim_{M\to\infty}\int_2^M (x-1)^{-1/3}\,\mathrm{d}x \\
&= \lim_{M\to\infty}\left[\frac{3}{2}(M-1)^{2/3}-\frac{3}{2}\right] = \infty,
\end{aligned}
$$

and so this improper integral **diverges**.

8. this is an improper integral since the interval of integration is $(-\infty,\infty)$ and hence is not a closed interval. We evaluate this improper integral by breaking it up as the sum of two improper integrals $\int_{-\infty}^{\infty} x\mathrm{d}x/(x^2+4) = \int_{-\infty}^{0} x\mathrm{d}x/(x^2+4) + \int_0^\infty x\mathrm{d}x/(x^2+4)$, and evaluating the two resulting improper integrals separately. So,

$$
\begin{aligned}
\int_{-\infty}^0 \frac{x}{x^2+4}\,\mathrm{d}x &= \lim_{M\to-\infty}\int_M^0 \frac{x}{x^2+4}\,\mathrm{d}x \\
&= \lim_{M\to-\infty}\left[\frac{1}{2}\ln(M^2+4)-\frac{1}{2}\ln(4)\right] = \infty.
\end{aligned}
$$

Since one of these two improper integrals diverges, we don't need to evaluate the other one, as the original improper integral $\int_{-\infty}^{0} x\mathrm{d}x/(x^2+4)$ necessarily **diverges**.

9. this is an improper integral, as the integrand is continuous on $(0,1]$ and $\lim_{x\to 0+} e^{\sqrt{x}}/\sqrt{x} = \infty$. So, we evaluate:

$$\begin{aligned} \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &= \lim_{c\to 0+} \int_c^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \\ &= \lim_{c\to 0+} (2 - 2\sqrt{c}) = 2, \end{aligned}$$

and so this improper integral **converges to** 2.

10. this is an improper integral, as the interval of integration is $[1,\infty)$ and so is not a closed interval. Moreover, the integrand is not continuous at 0 but $\lim_{x\to 1+} 1/x\ln(x) = \infty$, and so we need to break this improper integral into the sum of two improper integrals $\int_1^\infty \mathrm{d}x/x\ln(x) = \int_1^2 \mathrm{d}x/x\ln(x) + \int_2^\infty \mathrm{d}x/x\ln(x)$, and evaluate the two resulting improper integrals separately. So,

$$\begin{aligned} \int_1^2 \frac{1}{x\ln(x)}\,\mathrm{d}x &= \lim_{c\to 1+} \int_c^2 \frac{1}{x\ln(x)}\,\mathrm{d}x \\ &= \lim_{c\to 1+} (\ln(\ln(2)) - \ln(\ln(c))) = \infty, \end{aligned}$$

and so this improper integral diverges, and so the original improper integral $\int_1^\infty \mathrm{d}x/x\ln(x)$ necessarily **diverges**.