**Question**

Show that $\int_{-\infty}^{\infty}(1+x)\mathrm{d}x/(1+x^2)$ diverges, but that $\lim_{t\to\infty}\int_{-t}^{t}(1+x)\mathrm{d}x/(1+x^2) = \pi$.

**Answer**

We first need to write $\int_{-\infty}^{\infty}(1+x)\mathrm{d}x/(1+x^2)$ as the sum of two improper integrals, for instance

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{0} \frac{1+x}{1+x^2}\,\mathrm{d}x + \int_{0}^{\infty} \frac{1+x}{1+x^2}\,\mathrm{d}x,$$

and then evaluate the two resulting improper integrals separatedly. So,

$$\begin{aligned}
\int_{0}^{\infty} \frac{1+x}{1+x^2}\,\mathrm{d}x &= \lim_{M\to\infty}\int_{0}^{M} \frac{1+x}{1+x^2}\,\mathrm{d}x \\
&= \lim_{M\to\infty}\left[\int_{0}^{M} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{0}^{M} \frac{x}{1+x^2}\,\mathrm{d}x\right] \\
&= \lim_{M\to\infty}\left[(\arctan(M) - \arctan(0)) + \left(\frac{1}{2}\ln(1+M^2) - \frac{1}{2}\right)\right] = \infty,
\end{aligned}$$

since $\lim_{M\to\infty}\ln(1+M^2) = \infty$, and so the original improper integral $\int_{-\infty}^{\infty}(1+x)\mathrm{d}x/(1+x^2)$ diverges.

However, when we evaluate $\lim_{t\to\infty}\int_{-t}^{t}(1+x)\mathrm{d}x/(1+x^2)$, we get

$$\begin{aligned}
\lim_{t\to\infty}\int_{-t}^{t} \frac{1+x}{1+x^2}\,\mathrm{d}x &= \lim_{t\to\infty}\left[\int_{-t}^{t} \frac{1}{1+x^2}\,\mathrm{d}x + \int_{-t}^{t} \frac{x}{1+x^2}\,\mathrm{d}x\right] \\
&= \lim_{t\to\infty}\left[(\arctan(t) - \arctan(-t)) + \frac{1}{2}\left(\ln(1+t^2) - \ln(1+(-t)^2)\right)\right] \\
&= \lim_{t\to\infty} 2\arctan(t) = 2\frac{\pi}{2} = \pi,
\end{aligned}$$

and so $\lim_{t\to\infty}\int_{-t}^{t}(1+x)\mathrm{d}x/(1+x^2)$ converges. (Here, we use that $\arctan(-t) = -\arctan(t)$.)