Question

In a cotton spinning room dust particles are produced according to a Poisson process of rate λ. An extractor fan removes the dust particles at a rate μ times the number of particles present.

Let $p_n(t)$ denote the probability that there are n dust particles present at time t. Obtain the forward differential equations for $p_n(6)$. The generating function for these probabilities is defined by

$$G(z,\ t) = \sum_{n=0}^{\infty} p_n(t) z^n.$$

Show that

$$\frac{\partial G}{\partial t} + \mu(z-1)\frac{\partial G}{\partial z} = \lambda(z-1)G.$$

Verify that this equation is satisfied by

$$G(z,\ t) = \exp\left(\frac{\lambda z}{\mu}\right) \cdot f(e^{-\mu t}(z-1))$$

where f is an arbitrary differentiable function.

Find f and hence G, if the room is free of dust at time $t = 0$.

Answer

Let $X(t)$ denote the number of particles present at time t.

$n = 0$:

$$\begin{aligned} P_0(t+\delta t) &= P(X(t+\delta t) = 0 \mid X(t) = 0)p_0(t) \\ &\quad +P(X(t+\delta t) = 0 \mid X(t) = 1)p_1(t) \\ &= (1-\lambda\delta t)p_0(t) + (1-\lambda\delta t)\mu\delta t p_1(t) \\ &= (1-\lambda\delta t)p_0(t) + \mu\delta t p_1(t) \end{aligned}$$

so $p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$.

$n > 0$:

$$\begin{aligned} P_n(t+\delta t) &= P(X(t+\delta t) = n \mid X(t) = n)p_n(t) \\ &\quad +P(X(t+\delta t) = n \mid X(t) = n+1)p_{n+1}(t) \\ &\quad +P(X(t+\delta t) = n \mid X(t) = n-1)p_{n-1}(t) \\ &= [(1-\lambda\delta t)(1-n\mu\delta t) + \lambda\delta t n\mu\delta t]p_n(t) \\ &\quad +[(1-\lambda\delta t)(n+1)\mu\delta t]p_{n+1}(t) \\ &\quad [\lambda\delta t(1-(n-1)\mu)\delta t]p_{n-1}(t) \end{aligned}$$

$$P_n'(t) = 0(\lambda + n\mu)p_n(t) + (n+1)\mu p_{n+1}(t) + \lambda p_{n-1}(t)$$

Now let $G(z\,t) = \sum_{n=0}^{\infty} p_n(t) z^n$

$$
\begin{aligned}
\frac{\partial G}{\partial t} &= \sum_{n=0}^{\infty} p_n'(t) z^n \\
&= \sum_{n=0}^{\infty} [-(\lambda+\mu n) p_n(t) + (n+1)\mu p_{n+1}(t) + \lambda p_{n-1}(t)] z^n \\
&\quad \text{(letting p}_{-1}\text{(t)} \equiv 0) \\
&= \sum_{n=0}^{\infty} -\lambda p_n(t0 z^n - \mu \sum_{n=0}^{\infty} n p_n(t) z^n \\
&\quad + \mu \sum_{n=0}^{\infty} (n+1) p_{n+1}(t) z^n + \lambda \sum_{n=0}^{\infty} p_{n-1}(t) z^n \\
&= -\lambda \sum_{n=0}^{\infty} p_n(t) z^n - \mu z \sum_{n=0}^{\infty} p_n(t) n z^{n-1} \\
&\quad + \mu \sum_{n=1}^{\infty} p_n(t) n z^{n-1} + \lambda z \sum_{n=0}^{\infty} p_n(t) z^n \\
&= -\mu(z-1)\frac{\partial G}{\partial z} + \lambda(z-1)G
\end{aligned}
$$

If $G(z,\ t) = e^{\frac{\lambda z}{\mu}} f(e^{-\mu t}(z-1))$

$$\frac{\partial G}{\partial t} = e^{-\frac{\lambda z}{\mu}} f'(e^{-\mu t}(z-1)) \cdot (z-1) e^{-\mu t}(-\mu)$$

$$
\begin{aligned}
\frac{\partial G}{\partial z} &= \frac{\lambda}{\mu} e^{\frac{\lambda z}{\mu}} f(e^{-\mu t}(z-1)) \\
&\quad + e^{\frac{\lambda z}{\mu}} f'(e^{-\mu t}(z-1)) e^{-\mu t}
\end{aligned}
$$

$$\frac{\partial G}{\partial t} + \mu(z-1)\frac{\partial G}{\partial z} = \lambda(z-1) e^{\frac{\lambda z}{\mu}} f(e^{-\lambda t}(z-1)) = \lambda(z-1)G$$

Now if $X(0) = 0,\ G(z,\ 0) \equiv 1$

so $1 = e^{\frac{\lambda z}{\mu}} f(z-1)$

Thus $f(z-1) = e^{-\frac{\lambda z}{\mu}}$ i.e. $f(x) = e^{-\frac{\lambda(x+1)}{\mu}}$

so

$$
\begin{aligned}
G(z,\ t) &= \exp\left(\frac{\lambda z}{\mu}\right) \exp\left[-\frac{\lambda}{\mu}\left(e^{-\mu t}(z-1)+1\right)\right] \\
&= \exp\left[\frac{\lambda}{\mu}\left[z - e^{-\mu t}(z-1) - 1\right]\right] \\
&= \exp\left[\frac{\lambda}{\mu}(z-1)(1-e^{-\mu t})\right]
\end{aligned}
$$