Question

Define Lebesgue outer measure $m*$ in 3-dimensional Euclidean space $\mathbf{R}^3$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the transformation defined by

$$(x, y, z) \to (hx, ky, lz),$$

where h, k and l are nonzero real numbers. If E is a subset of $\mathbf{R}^3$, express $m^*(T(E))$ in terms of $m^*(E)$, proving your relationship from your dimension of m^*. Show also that if E is measurable then $T(E)$ is measurable.

Find the Lebesgue measure in $\mathbf{R}^3$ of the ellipsoid

$$\left\{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}.$$

[The Lebesgue measure of the unit ball in $\mathbf{R}^3$ may be assumed.]

Answer

If E is a subset of $\mathbf{R}^3$, we define Lebesgue outer measure m^* by

$$m^*(E) = \inf_{\{R_i\}} \sum_{i=1}^{\infty} |R_i|$$

Where $\{R_i\}$ denotes a system of rectangles of the from $\{(x_1, x_2, x_3) | a_i < x_1 < b_i \; i = 1, 2, 3\}$ with the property that $E \subseteq \bigcup_{i=1}^{\infty} R_i$, and where

$$|R_i| = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$$

Now if R_i is the rectangle defined by

$$\begin{array}{rcl} a_1 < & x_2 & < b_1 \\ a_2 < & x_2 & < b_2 \\ a_3 < & x_3 & < b_3 \end{array}$$

Then $T(R_i)$ is a rectangle defined by

$$A \left\{ \begin{array}{rcll} ha_1 < & x_2' & < hb_1 & (\text{or } hb_1 < x_1' < ha_1 \text{ if } h < 0) \\ ka_2 < & x_2' & < kb_2 & (\text{or } kb_1 < x_1' < ka_1 \text{ if } k < 0) \\ la_3 < & x_3' & < lb_3 & (\text{or } lb_1 < x_1' < la_1 \text{ if } l < 0) \end{array} \right.$$

Now if $\epsilon > 0$, we can find a system $\{R_i\}$, such that

$$\sum_{i=1}^{\infty} |R_i| \leq m^*(E) + \epsilon,$$

by the defination $T(R_i)$ is a rectangle, and since $\bigcup_{i=1}^{\infty} R_i \supseteq E$, we have $\bigcup_{i=1}^{\infty} T(R_i) \supseteq T(E)$, and so $m^*(T(E)) \leq \sum_{i=1}^{\infty} |T(R_i)|$

Now by equation A, we have

$$\begin{aligned} |T(R_i)| &= |h|(b_1 - a_1)|k|(b_2 - a_2)|l|(b_3 - a_3) \\ &= |hkl||r_i| \end{aligned}$$

Hence $m^*(T(E)) \leq |hkl| \sum_{i=1}^{\infty} |R_i| \leq |hkl|(m^*(E) + \epsilon)$

Thus

$$m^*(T(E)) \leq |hkl| m^*(E) \tag{1}$$

Now consideration of the inverse transformation T^{-1} applied to the set $T(E)$ shows similarly that

$m^*(E) = M^*(T^{-1}(T(E))) \leq \frac{1}{|hkl|} m^*(T(E))$

i.e.

$$m^*(T(E)) \geq |hkl| m^*(E) \tag{2}$$

Thus by (1) and (2),

$$m^*(T(E)) = |hkl| m^*(E)$$

If E is measurable, then for any set A, $m^*(A) = M^*(A - E) + m^*(A \cap E)$

Now let m B be an arbitrary subset of $\mathbf{R}^3$. Taking $A = T^{-1}(B)$

$$\begin{aligned} m^*(T^{-1}(B)) &= m^*(T^{-1}(B) - E) + m^*(T^{-1}(B) \cap E) \\ &= m^*(T^{-1}(B - T(E))) + m^*(T^{-1}(A \cap T(E))) \end{aligned}$$

Thus we have

$$\frac{1}{|hkl|} m^*(B) = \frac{1}{|hkl|} m^*(B - T(E)) + \frac{1}{|hkl|} m^*(B \cap T(E))$$

Hence, cancelling $\frac{1}{|hkl|}$, $T(E)$ is measurable.

Let $E = \left\{ (x, y, z) | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$

Let $S = \{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 1\}$

Let $T : (x, y, z) \to (|a|x, |b|y, |c|z)$

Then $T(S) = E$

Hence $m^*(E) = |abc| m^*(S)$

S and E are closed and so measurable. S so the unit sphere, so $m(S) = \frac{4}{3}\pi$

Hence $m(E) = \frac{4}{3}\pi |abc|$