Question

A satellite is initially near a planet, at a distance r_0 with perpendicular velocity v_0, where $r_0 v_0^2 < \frac{\mu}{m}$. Show that $\ddot{r}_0 < 0$ so that the orbit must have the form

$$\frac{l}{r} = 1 - e\cos\phi$$

if $\phi = 0$ is the initial line. If the planet has radius R what is the condition on R for the satellite to hit the planet? Where is the impact?

Answer

Newton's 2nd law: $\ddot{r} - r\dot{\phi}^2 = -\dfrac{\mu}{mr^2} \qquad r^2\dot{\phi} = h$

$h = r_0 v_0$ (as initially $\mathbf{v} = v_0\mathbf{e}_\phi, r = r_0$)

$\ddot{r} = r\dfrac{h^2}{r^4} - \dfrac{\mu}{mr^2} = \dfrac{r_0^2 v_0^2}{r_0^3} - \dfrac{\mu}{mr_0^2} = \dfrac{mr_0^2 v_0^2 - \mu}{mr_0^2} < 0$ as $mr_0 v_0^2 < \mu$

$\dfrac{l}{r} = 1 - e\cos\phi$ therefore $\dfrac{l}{r_0} = 1 - e\cos 0 \Rightarrow \dfrac{l}{r_0} = 1 - e$

The planet has radius R, thus the satellite hits at $\phi = \phi_0$,

when $\dfrac{l}{r} = 1 - e\cos\phi_0$.

This only has a solution if $\frac{l}{r} \leq 1 + e$, therefore $R \geq \dfrac{l}{1+e} = \dfrac{r_0(1-e)}{1+e}$.

Impact is at $\cos\phi_0 = \dfrac{1}{e}\left(1 - \dfrac{l}{R}\right)$

i.e., $\cos\phi_0 = \dfrac{1}{e}\left(1 - \dfrac{r_0(1-e)}{R}\right)$