Question

There are many reasons why $\mathbf{F} = -\frac{GmM}{r^2}\mathbf{e}_r$ is not exactly correct for a planet's orbit. Suppose that the correct force is

$$\mathbf{F} = -\frac{GmM}{r^2}\left(1 + \frac{k}{r}\right)\mathbf{e}_r,$$

with $\frac{k}{r} \ll 1$ (so that inverse square gravity is very nearly correct).

(a) Show that the equation for the orbit is

$$\frac{d^2u}{d\theta^2} + u = \left(\frac{\mu}{mh^2}\right)(1 + ku)\,.$$

(b) Solve this (linear) equation, and show that any bounded resulting orbit has apses separated by an angle $\pi + \frac{1}{2}\frac{\pi k\mu}{mh^2}$ approximately.

Answer

(a) book work: plug in for E

(b) Solve ordinary differential equation.

$$u = \frac{\mu}{mh^2\left(1 - \frac{k\mu}{mh^2}\right)} + A\cos\left\{\phi\sqrt{1 - \frac{\mu k}{mh^2}}\right\}$$

An apse occur when $\frac{du}{d\phi} = 0$

i.e.

$$\begin{aligned}\sin\left(\phi\sqrt{1 - \frac{k\mu}{mh^2}}\right) &= 0\\ \phi &= 0\,,\ \frac{\pi}{\sqrt{1 - \frac{\mu k}{mh^2}}}\quad \text{etc.}\end{aligned}$$

Now the angle between the apses is $\frac{\pi}{\sqrt{1 - \frac{\mu k}{mh^2}}} \approx \pi + \frac{1}{2}\frac{\mu k}{mh^2} + ...$