Question

For a planetary orbit the apses are defined as those points where $\frac{dr}{d\phi} = 0$.

(a) Show that $\frac{du}{d\phi} = 0$ at the apse of the orbit.

(b) At what points of the orbit is $\frac{d^2u}{d\phi^2} = 0$? Show that the acceleration term $\ddot{r} = 0$ there, and that at these points $\dot{\phi} = \left(\frac{\mu}{r^3}\right)^{\frac{1}{2}}$

Answer

(a) $\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \Rightarrow \frac{du}{d\phi} = 0$ at an apse

(b) $\frac{l}{r} = 1 + e\cos\phi$. Therefore $\frac{d^2u}{d\phi^2} = -\frac{e}{l}\cos\phi$

Therefore $\frac{d^2u}{d\phi^2} = 0$ at $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

Now $\dot{r} = -\frac{1}{u^2}\frac{du}{d\phi}\frac{d\phi}{dt}$.

Angular momentum $r^2\dot{\phi} = y \Rightarrow \dot{\phi} = hu^2$

Therefore $\dot{r} = -h\frac{du}{d\phi} \Rightarrow \ddot{r} = -h\frac{d^2u}{d\phi^2}\dot{\phi} = -h^2u^2\frac{d^2u}{d\phi^2}$

Thus $\frac{d^2u}{d\phi^2} \Leftrightarrow \ddot{r} = 0$

Using Newton's 2nd law; radial component: $\ddot{r} = r\dot{\phi} = -\frac{\mu}{r^2}$

At these points $\ddot{r} = 0 \Rightarrow \dot{\phi}^2 = \frac{\mu}{r^3}$