Question

Show that the equation of the elliptical orbit of a particle moving under the inverse square force central force $-\mu r^{-2}\mathbf{e}_r$ is

$$\frac{(x+ae)^2}{a^2}+\frac{y^2}{b^2}=1$$

where $a=-\dfrac{\mu}{2E}$ and $b=\dfrac{h}{\sqrt{\frac{-2E}{m}}}$.

Answer

$\dfrac{l}{r}=1+e\cos\phi.\quad 0<e<1$ for an ellipse

$$\begin{aligned}
l=r+e\cos\phi &= r+ex\\
r^2 &= (l+ex)^2\\
\Rightarrow x^2+y^2 &= l^2-2elx+e^2x^2\\
x^2(1-e)^2+2elx+y^2 &= l^2\\
(1-e^2)\left[\left(x+\frac{el}{1-e^2}\right)^2-\frac{e^2l^2}{(1-e^2)^2}\right]+y^2 &= l^2\\
\left(x+\frac{el}{1-e^2}\right)^2+\frac{y^2}{1-e^2} &= \frac{l^2}{(1-e^2)}\left[1+\frac{e^2}{1-e^2}\right]\\
&= \frac{l^2}{(1-e^2)^2}
\end{aligned}$$

Put $a=\dfrac{l}{1-e^2}$ then $\dfrac{(x+al)^2}{a^2}+\dfrac{y^2}{(1-e^2)a^2}=1$

Therefore $b=a\sqrt{1-e^2}$

Now from orbit theory $l=\dfrac{mh^2}{\mu}\qquad e=\sqrt{1+\dfrac{2mh^2E}{\mu^2}}$

Therefore $1-e^2=-\dfrac{2mh^2E}{\mu^2}$ Therefore $a=\dfrac{mh^2}{\mu}\,\dfrac{\mu}{-2mh^2E}=-\dfrac{\mu}{2E}$

So $b=-\dfrac{\mu}{2E}\sqrt{\dfrac{-2mh^2E}{\mu^2}}=\dfrac{h}{\sqrt{\frac{-2E}{m}}}$