Question
Describe the characteristic properties of the Poisson process. Let $X(t)$ denote the total number of events occurring in a time interval of length t in a compound Poisson process. Then $X(t)$ has probability generating function

$$G_t 9z) = \exp(\lambda t A(z) - \lambda t).$$

Explain what is meant by a compound Poisson process, and the meaning of λ and $A(z)$ in the formula for $G_t(z)$.

Coachloads of student visit the Hirst memorial museum every day. It is open from 8 a.m. to 6 p.m. Coaches arrive according to a Poisson process of rate λ an hour, and each coach carries a full load of n passengers. Each visitor decides independently with probability p to contribute 1 towards upkeep of the museum. Find the mean and variance of the total amount of money contributed each day.

Answer
Let $N(a,\ b)$ denote the number of events occurring in the time interval $a < t \leq b$. This is an integer-valued random variable. These random variables form a Poisson process if the following properties hold.

(i) Numbers of events in non-overlapping intervals are independent.

(ii) The probabilities of an event occurring in a small time interval is roughly proportional to its length.

i.e.

$P(N(t,\ t+\delta t) = 1) = \lambda\delta t + o(\delta t)$

$P(N(t,\ t+\delta t) = 0) = 1 - \lambda\delta t + o(\delta t)$

as $\delta t \to 0$.

(iii) λ is constant i.e. we have time homogeneity.

(i) and (ii) is equivalent to saying that the number of events occurring in a time interval of length t has a Poisson distribution with parameter λt.

Now suppose that

(a) points occur in a Poisson process with rate λ. Let $N(t)$ be the number of events in time t.

(b) Y_i events occur at each point, where the Y_i are i.i.d random variables

(c) $Y_1,\ Y_2, \cdots$ and $N(t)$ are independent.

The total number of events occurring in time t is

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

and is said to have a compound Poisson distribution. If each Y_i has probability generating function $A(z)$ then $X(t)$ has probability generating function

$$G_t(z) = \exp(\lambda t A(z) - \lambda t)$$

In this example the number of people in a coachload who decide to contribute will be a $B(n\,p)$ random variable, with probability generating function $(q + pz)^n$, where $q = 1 - p$.

The museum is open for 10 hours, and so the probability generating function for the total contribution is

$$G(z) = \exp(10\lambda(q + pz)^n - 10\lambda)$$

The mean is given by $G'(1)$ and the variance by $G''(1) + G'(1) - G'(1)^2$.

$$\begin{aligned}
G(z) &= \exp(-10\lambda) \cdot \exp(10\lambda(q+pz)^n) \\
G'(z) &= \exp(-10\lambda) \cdot \exp(10\lambda(q+pz)^n) \cdot 10\lambda np(q+pz)^{n-1} \\
\text{so } \mathrm{G}'(1) &= 10\lambda np \text{ since } \mathrm{p} + \mathrm{q} = 1 \\
G''(z) &= \exp(-10\lambda) \cdot \exp(10\lambda(q+pz)^n) \cdot 100\lambda^2 n^2 p^2 (q+pz)^{2(n-1)} \\
&\quad + \exp(-10\lambda)\exp(10\lambda(q+pz)^n) \cdot \\
&\quad\ 10\lambda n(n-1)p^2(q+pz)^{n-2} \\
G''(1) &= 100\lambda^2 n^2 p^2 + 10\lambda n(n-1)p^2 \\
\text{so} &\quad G''(1) + G'(1) - G'(1)^2 \\
&= 100\lambda^2 n^2 p^2 + 10\lambda n(n-1)p^2 + 10\lambda np - (10\lambda np)^2 \\
&= 10\lambda np((n-1)p + 1) \\
&= 10\lambda np(np + q)
\end{aligned}$$