Question

Now consider the Bessel function as both its order, n and argument x get large. By setting $x = nr$, show that

$$J_n(nr) \sim \sqrt{\frac{2}{\pi n}}(r^2 01)^{-\frac{1}{4}} \cos\left\{n\left[\sqrt{r^2-1} - \text{arccos}\left(\frac{1}{\text{r}}\right)\right] - \frac{\pi}{4}\right\}.$$

Answer

Start with the integral representation:

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,dt \\ &= \sum_{\pm} \frac{1}{2\pi}\int_0^\pi e^{\pm int}e^{\mp ix\sin t}\,dt \end{aligned}$$

<u>NB</u> and <u>now</u> $x \to +\infty,\ n \to +\infty$

We set $x = nr$ by hint of question.

$J_n(x) = \sum_{\pm} \dfrac{1}{2\pi}\displaystyle\int_0^\pi e^{\pm int \mp inr\sin t}\,dt$

so that $h^{(\pm)}(t) = \pm r\sin t \mp t$

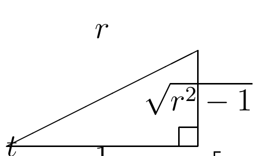
and we consider just $n \to +\infty,\ r$ fixed.

$$\text{Focus on}\quad \begin{aligned} h^{(+)}(t) &= +r\sin t - t \\ h^{(+)'}(t) &= r\cos t - 1 \\ h^{(+)''}(t) &= -r\sin t \end{aligned}$$

So stationary point at $t = \text{arccos}\left(\dfrac{1}{r}\right) \Rightarrow r \geq 1$ for real contribution

$$\Rightarrow \begin{aligned} h^{(+)}\left(\text{arccos}\left(\frac{1}{r}\right)\right) &= r\sin\left[\text{arccos}\left(\frac{1}{r}\right)\right] - \text{arccos}\left(\frac{1}{r}\right) \\ h^{(+)''}\left(\text{arccos}\left(\frac{1}{r}\right)\right) &= -r\sin\left[\text{arccos}\left(\frac{1}{r}\right)\right] \end{aligned}$$

This can be simplified by considering the right-angled triangle:



Therefore $h^{(+)}\left[\text{arccos}\left(\dfrac{1}{r}\right)\right] = \sqrt{r^2-1} - \text{arccos}\left(\dfrac{1}{r}\right)$

$h^{(+)''}\left[\text{arccos}\left(\dfrac{1}{r}\right)\right] = -\sqrt{r^2-1}$

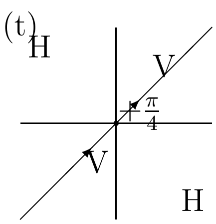
NB We assume $\underline{r > 1}$

Locally at stationary point we have

$$\begin{aligned} & -in\left[h^{(+)}(t) - h^{(+)}\left(\arccos\left(\frac{1}{r}\right)\right)\right] \\ = & -in\frac{(-\sqrt{r^2-1})}{2}\left(t - \arccos\left(\frac{1}{r}\right)\right)^2 \\ = & \frac{\in}{2}\sqrt{r^2-1}\left(t - \arccos\left(\frac{1}{r}\right)\right)^2 \quad r \geq 1 \end{aligned}$$

<u>Twists</u>:

Locally require



$$\begin{cases} Re[-in[\left(h^{(+)}(t) - h^{(+)}\left(\arccos\left(\frac{1}{r}\right)\right)\right] < 0 \\ Re\left[\frac{i\sqrt{r^2-1}}{2}\left(t - \arccos\left(\frac{1}{r}\right)\right)^2\right] < 0 \\ \left[t - \arccos\left(\frac{1}{r}\right)\right]^2 = Re^{i\theta_+} \\ \Rightarrow Re\left[\frac{i\sqrt{r^2-1}}{2}R^2 e^{2i\theta_+}\right] < 0 \\ \text{since r} \geq 1 \\ \Rightarrow Re\left[ie^{2i\theta_+}\right] < 0 \\ \Rightarrow -\sin 2\theta_+ < 0 \\ \sin 2\theta_+ > 0 \end{cases}$$

Angle of twist from:

$$\begin{aligned} Im\left\{in\left[h^{(+)}(t) - h^{(+)}\left(\arccos\frac{1}{r}\right)\right]\right\} &= 0 \\ \Rightarrow \cos 2\theta_+ &= 0 \\ \Rightarrow \theta_+ &= \underline{+\frac{\pi}{4}} \end{aligned}$$

PICTURE

Can use equation (3.99) with n playing the role of x.

$$
\begin{aligned}
J_+ &= \frac{1}{2\pi}\int_0^{\pi} e^{+in(t-r\sin t)}\,dt \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|nh^{(}+)''(t_0)|}}e^{-ing^{(}+)(t_0)+i\theta_+}f^{(}+)(t_0) \\
& \quad t_0 = \arccos\left(\frac{1}{r}\right) \\
& \quad f^{(}+)(t) = 1,\ \text{now since both exponentials contribute} \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|-n\sqrt{r^2-1}|}}e^{-in(\sqrt{r^2-1}-\arccos(\frac{1}{r}))+i\frac{\pi}{4}} \\
&\sim \sqrt{\frac{1}{2n\pi\sqrt{r^2-1}}}e^{-in(\sqrt{r^2-1}-\arccos(\frac{1}{r}))+i\frac{\pi}{4}}
\end{aligned}
$$

J_- is similar due to $+/-$ difference in $h^{(-)}(t)$.
$h^{(}-)\left(\arccos\left(\frac{1}{r}\right)\right) = -\sqrt{r^2-1} + \arccos\left(\frac{1}{r}\right)$
$h^{(}-)''\left(\arccos\left(\frac{1}{r}\right)\right) = +\sqrt{r^2-1} \quad (r > 1)$
$\theta_- = -\frac{\pi}{4}$
so using (3.99) with n playing the role of x:

$$
\begin{aligned}
J_- &= \frac{1}{2\pi}\int_0^{\pi} e^{-in(t-r\sin t)}\,dt \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|nh^{(}-)''(t_0)|}}e^{-inh^{(-)}(t_0)+i\theta-f^{(}-)(t_0)} \\
& \quad t_0 = \arccos\left(\frac{1}{r}\right),\ f^{(}-)(t) = 1 \\
&\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{|n\sqrt{r^2-1}|}}e^{-in(-\sqrt{r^2-1}+\arccos(\frac{1}{r}))-i\frac{\pi}{4}} \\
&\sim \frac{1}{\sqrt{2\pi n\sqrt{r^2-1}}}e^{+in(\sqrt{r^2-1}-\arccos(\frac{1}{r}))-\frac{i\pi}{4}}
\end{aligned}
$$

The total integral approximation is obtained by adding J_+ and J_-.

$$
J = J_+ + J_-
$$

$$\begin{aligned}
&\sim \frac{1}{\sqrt{2\pi n\sqrt{r^2-1}}}\left\{ \begin{array}{l} e^{+in(\sqrt{r^2-1}-\arccos(\frac{1}{r}))-\frac{\pi}{4}} \\ +e^{-in(\sqrt{r^2-1}-\arccos(\frac{1}{r}))+\frac{\pi}{4}} \end{array} \right\} \\
&\sim \sqrt{\frac{2}{\sqrt{\pi n\sqrt{r^2-1}}}}\cos\left(n\left[\sqrt{r^2-1}-\arccos\left(\frac{1}{r}\right)\right]-\frac{\pi}{4}\right) \\
&\sim \sqrt{\frac{2}{\pi n}}(r^2-1)^{-\frac{1}{4}}\cos\left(n\left[\sqrt{r^2-1}-\arccos\left(\frac{1}{r}\right)\right]-\frac{\pi}{4}\right)
\end{aligned}$$

as required. $n \to +\infty$