Question

Examine where the dominant contributions arises from, perform a local expansion and Use Watson's lemma to show

(a) $\displaystyle\int_0^\infty e^{-x(t^2+2t)}(1+t)^{\frac{5}{2}}\,dt \sim \frac{1}{2x},\ x \to +\infty$

(b) $\displaystyle\int_0^\infty e^{-x(t^2+2t)}\log(1+t)\,dt \sim \frac{\log 2}{2x},\ x \to +\infty$

(c) $\displaystyle\int_0^\infty e^{-x(t^2+2t)}\log(1+t)\,dt \sim \frac{1}{4x^2},\ x \to +\infty$

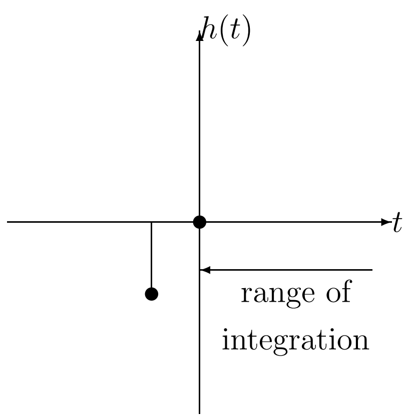
(d) $\displaystyle\int_0^\infty e^{-x(t^2+2t)}(t+3t^2)^{-\frac{1}{2}}\,dt \sim \sqrt{\frac{\pi}{2x}},\ x \to +\infty$

Answer

(a) $\displaystyle\int_0^\infty e^{-x(t^2+2t)}(1+t)^{\frac{5}{2}}\,dt \quad x \to +\infty$

$$h(t) = (t^2+2t) \Rightarrow \underbrace{h'(t) = 2t+2}_{} \Rightarrow h''(t) = 2$$

min. at $t = -1$ which is <u>outside</u> our range of integration



Thus the minimum value of $e^{-xh(t)}$ occurs when $t = 0,\ h(t) = 0$. The dominant contribution will come from this <u>linear</u> endpoint at $t = 0$. This differs slightly from the examples in the notes.

We could try an integration by parts but this looks messy. Try instead a Watson type argument and Taylor expand about $t = 0$.

$$h(t) - \underbrace{h(0)}_{=0} = \underbrace{h'(0)}(t-0) + O(t=0)^2 \quad (1)$$

Not zero here as it's a linear endpoint

$h'(0) = 2$ (from above)

Therefore set
$$\begin{aligned} u &= h(t) - h(0) \quad (2) \\ du &= h'(t)\,dt \quad (3) \end{aligned}$$

But (1) $\Rightarrow\ u \approx 2t$.

So in integral:

$$\begin{aligned} I &= \int_0^\infty e^{-x(t^2+2t)}(1+t)^{\frac{5}{2}}\,dt \\ &= e^{-x\ln(0)}\int_0^\infty e^{-xu}\frac{(1+t(u))^{\frac{5}{2}}}{h'(t(u))}\,du \\ &\approx \int_0^\infty e^{-xu}\frac{(1+\frac{u}{2})}{h'(t(u))}\,du \end{aligned}$$

$$h'(t) = \underbrace{h'(0)}_{\neq 0 \text{ as it's a linear endpoint}} + \frac{h''(0)}{2}(t-0)^2 + \cdots = 2$$

from above, to leading order.

Therefore $du \approx 2\,dt$

Therefore $I \approx \displaystyle\int_0^\infty e^{-xu}\frac{(1+\frac{u}{2})^{\frac{5}{2}}}{2}\,du$

Now apply Laplace: contribution centred about $u=0$ as $x \to +\infty$.

$$I \sim \underbrace{\frac{(1+\frac{0}{2})^{\frac{5}{2}}}{2}}\int_0^\infty e^{-xu}\,du, \text{ as x} \to +\infty$$

to leading order this is a constant.
So take it outside the integral

$$\sim \frac{1}{2x} \quad x \to +\infty \quad \text{as required}$$

(b) The dominant contribution again comes from $t = 0$ (same $h(t)$ as above). The only difference is the value of $f9t) = \log(2+t)$ at $t = 0$. Thus the method goes through as for (a) with:

$$\begin{aligned} I &= \int_0^\infty e^{-x(t^2+2t)} \log(2+t)\, dt \\ &\sim \frac{\log(2+0)}{2} \int_0^\infty e^{-xu}\, du \text{ as x} \to +\infty \\ &\sim \frac{\log 2}{2x} \quad x \to +\infty \end{aligned}$$

(c) Here the dominant contribution is again from $t = 0$ $(h(t) = t^2 + 2t$ again). But now $f(t) = \log(1+t)$ which is 0 at $t = 0$. This does not necessarily mean that the contribution from $t = 0$ vanishes. Instead we must go to higher order in the expansion of $\dfrac{f(t)}{h'(t)}$, keeping it inside the integral.

Proceed as above until:

$I = \int_0^\infty e^{-x(t^2+2t)} \log(1=t)\, dt - \int_0^\infty \dfrac{\log(1+t(u))}{h'(t(u))}\, du$

where $h'(t) \approx 2$ and $u \approx 2t$.

Now just expand the log inside the integral:

$I \sim \int_0^\infty \dfrac{e^{-xu}(t(u) - \frac{t^2(u)}{2} + \cdots)}{2}\, du \sim \int_0^\infty e^{-xu} \dfrac{u}{4}$ to leading order

$x \to +\infty$

Therefore $I \sim \dfrac{1}{4} \int_0^\infty e^{-xu}\, u \sim \dfrac{1}{4x^2} \quad x \to +\infty$

(d) As above the dominant contribution is from the linear $t = 0$ endpoint $(h(t) = t^2 + 2t)$.

Consider $f(t)$

$$f(t) = \frac{1}{\sqrt{t(1+3t)}} = \frac{1}{\sqrt{t}} + O(t^{\frac{1}{2}}),\ t \to o^+$$

The method proceeds as above, but we now retain the leading order of $f(t)$ as $t \to 0^+$ in the integral.

$$J = \int_0^\infty e^{-x(t^2+2t)}(t+3t^2)^{-\frac{1}{2}}\,dt = \int_0^\infty e^{-xu}\frac{[t(u)+3t^2(u)]^{-\frac{1}{2}}}{h'(t(u))}\,du$$

$h'(t) \approx 2$ to leading order and $u \approx 2t$

Thus $J \sim \int_0^\infty \frac{e^{-xu}}{2}\cdot\frac{1}{\sqrt{t(u)}} \approx \frac{1}{2}\int_0^\infty \frac{e^{-xu}}{\sqrt{\frac{u}{2}}}\,du = \frac{1}{\sqrt{2}}\int_0^\infty \frac{e^{-xu}}{u^{\frac{1}{2}}}\,du$

Remembering the definition of the Γ-function, this last integral is $\frac{\Gamma(\frac{1}{2})}{x^{\frac{1}{2}}}$.

Therefore $\int_0^\infty e^{-x(t^2+2t)}(t+3t^2)\,dt \sim \frac{\Gamma(\frac{1}{2})}{\sqrt{2x}} = \sqrt{\frac{\pi}{2x}} \quad x \to +\infty$