Question

By using the change of variables $\tau = \log t$ and integrating by parts, show that as$x \to +\infty$

(a) $\displaystyle\int_x^\infty \frac{dt}{t^2 \log t} \sim \frac{1}{x \log x}$

(b) $\displaystyle\int_2^x \frac{dt}{t \log(\log(t))} \sim \frac{\log x}{\log(\log(x))}$

Answer

(a) Let $\tau = \ln t \Rightarrow d\tau = \dfrac{dt}{t}$

Therefore $\displaystyle\int_x^\infty \frac{dt}{t^2 \ln t} = \int_x^\infty \frac{d\tau}{t \ln t} = \int_{\ln x}^\infty \frac{e^{-\tau}}{\tau}\, dt$

$x \to +infty \Rightarrow \ln x \to +\infty$

Use integration by parts

$dv = e^{-\tau} \quad u = \dfrac{1}{\tau}$

$v = -e^{-\tau} \quad du = -\dfrac{1}{\tau^2}$

Therefore

$$\begin{aligned}\int_x^\infty \frac{dt}{\tau^2 \ln t} &= \left[\frac{-e^{-\tau}}{\tau}\right]_{\ln x}^\infty - \int_{\ln x}^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau \\ &= \frac{e^{-\ln x}}{\ln x} + R \\ &= \frac{1}{x \ln x} + R\end{aligned}$$

$$\begin{aligned}|R| &= \left|\int_{\ln x}^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau\right| \\ &\leq \int_{\ln x}^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau \\ &= \frac{1}{(\ln x)^2}\int_{\ln x}^\infty \left(\frac{\ln x}{\tau}\right)^2 e^{-\tau}\, d\tau \\ &\quad \ln x > \tau \\ &\leq \frac{1}{(\ln x)^2}\int_{\ln x}^\infty e^{-\tau}\, d\tau\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x(\ln x)^2} \\
&= o\left(\frac{1}{x\ln x}\right) \text{ since } \ln \mathrm{x} \geq 1,\ \mathrm{x} \to +\infty
\end{aligned}
$$

Therefore $\displaystyle\int_x^\infty \sim \frac{1}{x\ln x},\ x \to +\infty$

(b) This is more involves: $\tau = \ln t$ as above gives

$$\int_2^x \frac{dt}{t\log(\log(t))} = \int_{\ln 2}^{\ln x} \frac{d\tau}{\log\tau}$$

set $\tau_1 = \log\tau \Rightarrow d\tau_1 = \dfrac{d\tau}{\tau}$

Therefore $K = \displaystyle\int_2^x \frac{dt}{t\log(\log(t))} = \int_{\log(\log(2))}\log(\log(x))\frac{d\tau_1 e^{\tau_1}}{\tau_1}$

Now integrate by parts:

$$v = \frac{1}{\tau_1} \qquad du = e^{\tau_1}$$

$$dv = -\frac{1}{tau_1^2} \qquad u = e^{\tau_1}$$

$$\begin{aligned} K &= \left[\frac{e^{\tau_1}}{\tau_1}\right]_{\log(\log(2))}^{\log(\log(x))} + \int_{\log(\log(2))}^{\log(\log(x))} \frac{e^{\tau_1}}{\tau_1^2}\,d\tau_1 \\ &= \frac{e^{\log(\log(x))}}{\log(\log(x))} - \frac{e^{\log(\log(2))}}{\log(\log(2))} + \int_{\log(\log(2))}^{\log(\log(x))} \frac{e^{\tau_1}}{\tau_1^2}\,d\tau_1 \\ &= \frac{\log x}{\log(\log(x))} - \underbrace{\log 2\log(\log(2))}_{\substack{\text{just a} \\ \text{finite number}}} + \underbrace{O\left(\frac{\log x}{[\log(\log(x))]^2}\right)}_{\substack{\text{by similar arguments to} \\ \text{part (a)}}} \end{aligned}$$

Now as

$$\begin{aligned} x &\to +\infty \\ \log x &\to +\infty \\ \log(\log(x)) &\to +\infty \text{ (more slowly)} \end{aligned}$$

even so we still have

$$K = \frac{\log x}{\log(\log x)} + o\left(\frac{\log x}{\log(\log x)}\right)$$

so $K \sim \dfrac{\log x}{\log(\log x)}, \quad x \to +\infty$