QUESTION

If $f(t)$ is an integrable function on $[0,t]$ and $W(t)$ Brownian show that the following integration by parts formula holds:

$$\int_0^t f(t)\,dW = f(t)W(t) - \int_0^t W\,df$$

ANSWER

Consider $g = f(t)w(t)$

$$\begin{aligned}
\text{Itô} \Rightarrow dg &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial w}dw + \frac{1}{2}\frac{\partial^2 g}{\partial w^2}(dw)^2 \\
d(fw) &= \frac{\partial}{\partial t}(fw)dt + \frac{\partial g}{\partial w}(fw)dw + \frac{1}{2}\frac{\partial^2}{\partial w^2}(fw)dt \\
d(fw) &= w\frac{df}{dt}dt + fdw + 0 \\
\Rightarrow d(fw) &= wdf + fdw \\
\text{or } \int_0^t f\,dw &= \int_0^t d(wf) - \int_0^t w\,df \\
\Rightarrow \int_0^t f(t)\,dw &= f(t)w - \int_0^t w\,df
\end{aligned}$$

Since w is brownian and $w(0) = 0$.