QUESTION

Show that if $W(t)$ is a Brownian motion, then Itô's lemma implies the following results:

(i) $\int_0^t W^2\, dw = \frac{1}{3}W^3 - \int_0^t W\, dt$

(ii) $\int_0^t t\, dW = tW - \int_0^t W\, dt$

ANSWER

Itô's lemma for $f(x,t)$

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \dots$$

Let $x = w$ so that $dx = dw,\ (dx)^2 = (dw)^2 = dt$

(i) Consider $f = \frac{1}{3}w^3$

$$\begin{aligned} d(\frac{1}{3}w^3) &= 0.dt + \frac{\partial}{\partial w}(\frac{1}{3}w^3)dw + \frac{1}{2}\frac{\partial^2}{\partial w^2}(\frac{1}{3}w^3)dt \\ &= w^2 dw + w dt \end{aligned}$$

Therefore

$$\int d(\frac{1}{3}w^3) = \int w^2 dw + w dt$$

$$\Rightarrow \frac{1}{3}w^3 = \int_0^t w^2\, dw + \int_0^t w\, ds$$

Since $w(0) = 0$ as it's Brownian.

Or

$$\int_0^t w^2\, dw = \frac{1}{3}w^3 - \int_0^t x\, ds$$

(ii) Consider $f = tw$

$$\begin{aligned} df = d(tw) &= \frac{\partial}{\partial t}(tw) + \frac{\partial}{\partial w}(tw)dw + \frac{1}{2}\frac{\partial^2}{\partial w}(tw)(dw)^2 \\ &= wdt + tdw + 0 \\ \Rightarrow \int d(tw) &= \int w\, dt + \int t\, dw \\ \Rightarrow tw &= \int_0^t w\, ds + \int_0^t s\, dw \end{aligned}$$

Since $w(0) = 0$ when $t - 0$,

Or

$$\int_0^t s\,dw = tw - \int_0^t w\,ds$$